The numbers w_n have been studied by Moser and Wyman [6]. From the differential equation $W'(y) = -(1+y)W(y)$, we obtain the recurrence

$$w_{n+1} = -(w_n + nw_{n-1}),$$

from which the w_n are easily computed. The first few instances of (18) are

$$t_{m+1} - t_m = mt_{m-1}$$
$$t_{m+2} - 2t_{m+1} = 2\binom{m}{2}t_{m-2}$$
$$t_{m+3} - 3t_{m+2} + 2t_m = 6\binom{m}{3}t_{m-3}$$
$$t_{m+4} - 4t_{m+3} + 8t_{m+1} - 2t_m = 24\binom{m}{4}t_{m-4}.$$

A natural question is: To what series does this method apply? In other words, we want to characterize those Hurwitz series $f(x)$ for which there exist Hurwitz series $h(z)$ and $g(z)$, with $h(0) = 1$, $g(0) = 0$, and $g'(0) = 1$, such that for all m, $n \geq 0$, the coefficient of $(x^m/m!)z^n$ in $h(z)f[x + g(z)]$ is integral.

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A QUADRATIC PROPERTY OF CERTAIN LINEARLY RECURRENT SEQUENCES

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In [1] one of the authors proved the following result.

Let u be a real number such that $u > 1$, and let $\{x_n\}_{n\geq 0}$ be a sequence of nonnegative real numbers such that

$$x_{n+1} = ux_n + \sqrt{(u^2 - 1)(x_n^2 - x_0^2) + (x_1 - ux_0)^2}$$

for every $n \geq 0$. Then

$$x_{n+2} = 2ux_{n+1} - x_n$$

for every $n \geq 0$; and, in particular, if u, x_0, x_1 are integers, then x_n is an integer for every $n \geq 0$.

In this note we shall show that, under certain conditions, the preceding result admits a converse.

We begin with the following general preliminary proposition.

Proposition: Let R be a commutative ring with unit element; let $t, u \in R$, and define a polynomial $f \in R[X, Y]$ by

$$f(X, Y) = tX^2 - 2uXY + Y^2.$$

If $\{r_n\}_{n \geq 0}$ is a sequence of elements of R such that

$$r_{n+2} = 2ur_{n+1} - tr_n$$

for every $n \geq 0$, then

$$f(r_n, r_{n+1}) = t^n f(r_0, r_1)$$

for every $n \geq 0$.

Proof: We shall prove this result by induction. The conclusion holds identically for $n = 0$. Assume now that it holds for some $n \geq 0$. Then

$$\begin{aligned} f(r_{n+1}, r_{n+2}) &= tr_{n+1}^2 - 2ur_{n+1}r_{n+2} + r_{n+2}^2 \\ &= tr_{n+1}^2 - 2ur_{n+1}(2ur_{n+1} - tr_n) + (2ur_{n+1} - tr_n)^2 \\ &= t(tr_n^2 - 2ur_n r_{n+1} + r_{n+1}^2) \\ &= tf(r_n, r_{n+1}) \\ &= tt^n f(r_0, r_1) \\ &= t^{n+1} f(r_0, r_1), \end{aligned}$$

which shows that the conclusion also holds for $n + 1$. ■

This proposition can be applied in some familiar particular cases:

If we take

$$R = Q, \ t = -1, \ u = 1/2,$$

we find that the Fibonacci and Lucas sequences satisfy

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$$
$$L_{n+1}^2 - L_{n+1}L_n - L_n^2 = 5(-1)^{n+1}$$

for every $n \geq 0$.

And if we take

$$R = Z, \ t = -1, \ u = 1,$$

we also find that the Pell sequence satisfies

$$P_{n+1}^2 - 2P_{n+1}P_n - P_n^2 = (-1)^n$$

for every $n \geq 0$.

We are now in a position to state and prove our result.

Theorem: Let t, u be real numbers such that

$$t^2 = 1 \quad \text{and} \quad u > \max(t, 0),$$

and let $\{x_n\}_{n \geq 0}$ be a sequence of real numbers such that

$$x_1 \geq \max(ux_0, (t/u)x_0, 0)$$

and satisfying

$$x_{n+2} = 2ux_{n+1} - tx_n$$

for every $n \geq 0$. We then have:

(i) $ux_{n+1} \geq \max(tx_n, 0)$ for every $n \geq 0$; and

(ii) $x_{n+1} = ux_n + \sqrt{(u^2 - t)(x_n^2 - t^n x_0^2) + t^n(x_1 - ux_0)^2}$ for every $n \geq 0$.

Proof: We shall prove (i) by induction. Our assumptions clearly imply that the stated inequality holds when $n = 0$. Now suppose that $ux_{n+1} \geq \max(tx_n, 0)$ for some $n \geq 0$. As the given conditions on t, u imply that $u > 0$ and $u^2 > t$, we first deduce that $x_{n+1} \geq 0$, and then that

$$\begin{aligned} ux_{n+2} &= u(2ux_{n+1} - tx_n) \\ &= u^2x_{n+1} + u(ux_{n+1} - tx_n) \geq u^2x_{n+1} \geq \max(tx_{n+1}, 0), \end{aligned}$$

as required.

Since $x_1 - ux_0 \geq 0$, it is clear that, in order to prove (ii), we need only consider the case where $n > 0$. In view of the proposition, we have

$$tx_{n-1}^2 - 2ux_{n-1}x_n + x_n^2 = t^{n-1}(tx_0^2 - 2ux_0x_1 + x_1^2).$$

Since $t^2 = 1$, we also have

$$x_{n-1}^2 - 2tux_{n-1}x_n + tx_n^2 = t^n(tx_0^2 - 2ux_0x_1 + x_1^2),$$

and hence

$$-2tux_{n-1}x_n + x_{n-1}^2 = -tx_n^2 + t^{n+1}x_0^2 - 2t^nux_0x_1 + t^nx_1^2;$$

it then follows that

$$\begin{aligned} (ux_n - tx_{n-1})^2 &= u^2x_n^2 - 2tux_{n-1}x_n + x_{n-1}^2 \\ &= u^2x_n^2 - tx_n^2 + t^{n+1}x_0^2 + t^nx_1^2 - 2t^nux_0x_1 \\ &= (u^2 - t)(x_n^2 - t^nx_0^2) + t^n(x_1 - ux_0)^2. \end{aligned}$$

By virtue of (i), we now conclude that

$$\begin{aligned} x_{n+1} &= 2ux_n - tx_{n-1} \\ &= ux_n + (ux_n - tx_{n-1}) \\ &= ux_n + \sqrt{(ux_n - tx_{n-1})^2} \\ &= ux_n + \sqrt{(u^2 - t)(x_n^2 - t^nx_0^2) + t^n(x_1 - ux_0)^2}, \end{aligned}$$

which is what was needed. ∎

Applying this theorem to the three special sequences considered above, we obtain the following formulas for every $n \geq 0$:

$$\begin{aligned} F_{n+1} &= \tfrac{1}{2}(F_n + \sqrt{5F_n^2 + 4(-1)^n}) \\ L_{n+1} &= \tfrac{1}{2}(L_n + \sqrt{5L_n^2 + 20(-1)^{n+1}}) \\ P_{n+1} &= P_n + \sqrt{2P_n^2 + (-1)^n}. \end{aligned}$$

These formulas, of course, can also be derived directly from the quadratic equalities established previously.

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