ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, NM 87131

Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-448 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} F_{5i+1}L_{5i} \equiv 0 \pmod 5.$$

B-449 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove that, for all positive integers t,

$$\sum_{i=1}^{2t} (-1)^{i+1}F_{8i+1}L_{8i} \equiv 0 \pmod 7.$$

B-450 *Proposed by Lawrence Somer, Washington, D.C.*

Let the sequence $\{H_n\}_{n=0}^{\infty}$ be defined by $H_n = F_{2n} + F_{2n+2}$.

(a) Show that 5 is a quadratic residue modulo H_n for $n \geq 0$.

(b) Does H_n satisfy a recursion relation of the form $H_{n+2} = cH_{n+1} + dH_n$, with c and d constants? If so, what is the relation?

B-451 *Proposed by Keats A. Pullen, Jr., Kingsville, MD*

Let k, m, and p be positive integers with p an odd prime. Show that in base $2p$ the units digit of $m^{k(p-1)+1}$ is the same as the units digit of m.

B-452 *Proposed by P. L. Mana, Albuquerque, NM*

Let $c_0 + c_1x + c_2x^2 + \cdots$ be the Maclaurin expansion for $[1-ax)(1-bx)]^{-1}$, where $a \neq b$. Find the rational function whose Maclaurin expansion is

$$c_0^2 + c_1^2x + c_2^2x^2 + \cdots$$

and use this to obtain the generating functions for F_n^2 and L_n^2.

B-453 *Proposed by Paul S. Bruckman, Concord, CA*

Solve in integers r, s, t with $0 \leq r < s < t$ the Fifibonacci Diophantine equation

$$F_{F_r} + F_{F_s} = F_{F_t}$$

and the analogous Lulucas equation in which each F is replaced by an L.

SOLUTIONS

Counting Hands

B-424 *Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, NM*

Of the $\binom{52}{5}$ possible 5-card poker hands, how many form a:

(i) full house?
(ii) flush?
(iii) straight?

Solution by Paul S. Bruckman, Concord, CA

(i) The two denominations represented in a full house may be chosen in $2\binom{13}{2}$ ways, the coefficient "2" reflecting the fact that the three-of-a-kind may appear in either of two ways. The individual cards for these denominations can be chosen in $\binom{4}{3}\binom{4}{2}$ ways. Thus, the total number of possible full houses is $2\binom{13}{2}\binom{4}{3}\binom{4}{2} = 3{,}744$.

(ii) The suit represented in a flush may be chosen in 4 ways, and the 5 cards of the flush in that suit may be chosen in $\binom{13}{5}$ ways. Hence, the total number of possible flushes (including "straight flushes") is $4\binom{13}{5} = 5{,}148$.

(iii) With the ace being either high or low, there are 10 different ways to choose the denominations appearing in a straight. With each of these ways, there are 4^5 choices for the individual cards. Thus, the total number of possible straights (including "straight flushes") is $10 \cdot 4^5 = 10{,}240$.

<u>*NOTE:*</u> Since the total number of possible straight flushes is $10 \cdot 4 = 40$, the answers to (ii) and (iii) above *excluding* straight flushes would be reduced by 40, and so would equal 5,108 and 10,200, respectively.

Also solved by John W. Milsom, Bob Prielipp, Charles B. Shields, Lawrence Somer, Gregory Wulczyn, and the proposer.

Average in a Fixed Rank

B-425 *Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, NM*

Let k and n be positive integers with $k < n$, and let S consist of all k-tuples $X = (x_1, x_2, \ldots, x_k)$ with each x_j an integer and

$$1 \leq x_1 < x_2 < \cdots < x_k \leq n.$$

For $j = 1, 2, \ldots, k$, find the average value $\overline{x}_j$ of x_j over all X in S.

Solution by Graham Lord, Université Laval, Québec, Canada

The number of k-tuples X in which $x_j = m$ is $\binom{m-1}{j-1}\binom{n-m}{k-j}$. [Choose the $j-1$ smaller integers from among the first $m-1$ natural numbers and the $k-j$ larger ones from among $m+1, \ldots, n$.] Evidently the total number of k-tuples,

$$\sum_{m=j}^{n-k+j} \binom{m-1}{j-1}\binom{n-m}{k-j},$$

is simply the number of k-subsets from $\{1, 2, \ldots, n\}$, that is $\binom{n}{k}$. Hence,

$$\bar{x}_j = \left\{\sum_{m=j}^{n-k+j} m\binom{m-1}{j-1}\binom{n-m}{k-j}\right\}\Big/\binom{n}{k}$$

$$= \left\{j\sum_{m+1=j+1}^{n-k+j+1}\binom{m+1-1}{j+1-1}\binom{n+1-(m+1)}{k+1-(j+1)}\right\}\Big/\binom{n}{k}$$

$$= j\cdot\binom{n+1}{k+1}\Big/\binom{n}{k}$$

$$= \frac{j(n+1)}{k+1}.$$

Also solved by Paul S. Bruckman and the proposer.

Fibonacci Pythagorean Triples

B-426 *Proposed by Herta T. Frietag, Roanoke, VA*

Is $(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2$ a perfect square for all positive integers n, i.e., are there integers c_n such that $(F_nF_{n+3}, 2F_{n+1}F_{n+2}, c_n)$ is always a Pythagorean triple?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

The answer to the question posed above is "yes" and $c_n = F_{2n+3}$. To establish this result, we observe that

$$F_nF_{n+3} = (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}) = F_{n+2}^2 - F_{n+1}^2$$

so

$$(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = (F_{n+2}^2 - F_{n+1}^2)^2 + 4F_{n+2}^2F_{n+1}^2$$
$$= (F_{n+2}^2 + F_{n+1}^2)^2 = F_{2n+3}^2.$$

[The last equality follows from the fact that $F_{n+1}^2 + F_n^2 = F_{2n+1}$ for each nonnegative integer n.]

Also solved by Paul S. Bruckman, M. J. DeLeon, A. F. Horadam, Graham Lord, John W. Milsom, A. G. Shannon, Charles B. Shields, Sahib Singh, Lawrence Somer, M. Wachtel, Gregory Wulczyn, and the proposer.

NOTE: Each of Horadam and Shannon pointed out that both B-402 and B-426 are special cases of general equation (2.2)′ in A. F. Horadam: "Special Properties of the Sequence $W(a, b, p, q)$," *The Fibonacci Quarterly* 5 (1967):425.

Closed Form, Ingeniously

B-427 *Proposed by Phil Mana, Albuquerque, NM*

Establish a closed form for $\sum_{k=1}^{n} k\binom{k}{2}\binom{n-k}{3}$.

Solution by Graham Lord, Université Laval, Québec, Canada

$$\text{The sum} = \sum_{k=2}^{n-3}(k+1)\binom{k}{2}\binom{n-k}{3} - \sum_{k=2}^{n-3}\binom{k}{2}\binom{n-k}{3}$$

$$= 3\cdot\sum_{k=2}^{n-3}\binom{k+1}{3}\binom{n-k}{3} - \binom{n+1}{6}$$

(continued)

$$= 3 \cdot \binom{n+2}{7} - \binom{n+1}{6} = \frac{3n-1}{7} \cdot \binom{n+1}{6}.$$

NOTE: As shown in my solution to B-425, $\sum_m \binom{m}{a}\binom{n-m}{b}$ counts the number of subsets of $a + b + 1$ elements chosen from a set of $n+1$ elements: this latter sum equals $\binom{n+1}{a+b+1}$.

Also solved by Paul S. Bruckman, Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

Closed Form, Industriously

B-428 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For odd positive integers w, establish a closed form for

$$\sum_{k=0}^{2s+1} \binom{2s+1}{k} F_{n+kw}^2 .$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

Since $F_j = (a^j - b^j)/\sqrt{5}$ [where $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$],

$$\sum_{k=0}^{2s+1} \binom{2s+1}{k} F_{n+kw}^2 = \frac{1}{5}\sum_{k=0}^{2s+1} \binom{2s+1}{k} (a^{2n+2kw} - 2(-1)^{n+kw} + b^{2n+2kw})$$

$$= \frac{1}{5}[a^{2n}(1+a^{2w})^{2s+1} - 2(-1)^n(1+(-1)^w)^{2s+1} + b^{2n}(1+b^{2w})^{2s+1}] \quad \text{(by the Binomial Theorem)}$$

$$= \frac{1}{5}[a^{2n}(1+a^{2w})^{2s+1} + b^{2n}(1+b^{2w})^{2s+1}] \quad \text{(because } w \text{ is odd)}$$

$$= \frac{1}{5}[a^{2n+(2s+1)w}(a^{-w}+a^w)^{2s+1} + b^{2n+(2s+1)w}(b^{-w}+b^w)^{2s+1}]$$

$$= \frac{1}{5}[a^{2n+(2s+1)w}(a^w - b^w)^{2s+1} + b^{2n+(2s+1)w}(b^w - a^w)^{2s+1}]$$

$$= \frac{1}{5}[(a^{2n+(2s+1)w} - b^{2n+(2s+1)w})(a^w - b^w)^{2s+1}]$$

$$= 5^s F_{2n+(2s+1)w} F_w^{2s+1}.$$

Also solved by Paul S. Bruckman, A. G. Shannon, and the proposer.

Yes, When Boiled Down

B-429 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Is the function

$$F_{n+10r}^4 + F_n^4 - (L_{8r} + L_{4r} - 1)(F_{n+8r}^4 + F_{n+2r}^4) + (L_{12r} - L_{8r} + 2)(F_{n+6r}^4 + F_{n+4r}^4)$$

independent of n? Here n and r are integers.

Solution by Paul S. Bruckman, Concord, CA and
Sahib Singh, Clarion State College, Clarion, PA, independently

Yes. It boils down to

$$12F_{2r}^2F_{4r}^2$$

or to

$$12(L_{12r} - 2L_{8r} - L_{4r} + 4)/25.$$

(The steps were deleted by the Elementary Problems editor.)

Also solved by Bob Prielipp and the proposer.

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