Give an element $i = (i_1, \ldots, i_k) \in \mathbb{Z}^k$ weight $w(i) = x_1^{i_1} \ldots x_k^{i_k}$ and define the weight $w(S)$ of $S \subseteq \mathbb{Z}^k$ to be the sum of the weights of the elements of S. The main result is that

$$w(H) + w(H + D)z + w(H + D + D)z^2 + \cdots \tag{33}$$

is a rational function in $x_1, \ldots, x_k$ and z. A consequence of this is that the sequence of volumes $(|H|, |H + D|, |H + D + D|, \ldots)$ forms a rational sequence.

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ON THE PROBABILITY THAT n AND $\Omega(n)$ ARE RELATIVELY PRIME

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To the memory of V. E. Hoggatt Jr.—my teacher and friend

It is a well-known result due to Chebychev that if n and m are randomly chosen positive integers, then $(n, m) = 1$ with probability $6/\pi^2$. It is the purpose of this note to show that if $\Omega(n)$ is the number of prime factors of n counted with multiplicity, then the probability that $(n, \Omega(n)) = 1$ is also $6/\pi^2$. Thus, as far as common factors are concerned, $\Omega(n)$ behaves randomly with respect to n.

Results of this type for fairly general additive functions have been proved by Hall [2], and in [1] and [3] he looks closely at the situation regarding the special additive function $g(n)$, the sum of the distinct prime factors of n. Hall's results do not apply to either $\Omega(n)$ or $\omega(n)$, the number of distinct prime factors of n, and so our result is of interest. Our proof, which is of an analytic nature, proceeds along classical lines, and so must surely be known to specialists in the field. In any case, it never seems to have been stated in the literature and so we felt it was worthwhile to prove it, particularly since it is interesting when viewed in the context of several celebrated results on the distribution of $\Omega(n)$ (see [5]) such as those of Hardy-Ramanujan and Erdös-Kac. By a slight modification of our proof, the same result can be established for $\omega(n)$; we have concentrated on $\Omega(n)$ for the sake of simplicity. Throughout, implicit constants are absolute unless otherwise indicated and p always denotes a prime number.

Theorem: Let

$$Q(x) = \sum_{1 \le n \le x,\ (n,\ \Omega(n)) = 1} 1 .$$

Then

$$Q(x) = \frac{6x}{\pi^2} + O(x(\text{logloglog } x)^{-1/3} \cdot (\text{loglogloglog } x)^{-1}).$$

To prove the theorem, we need a few auxiliary results.

Lemma 1: Let $x > 20$, and k be a positive integer such that

$$k \le \{\text{loglog } x/\text{logloglog } x\}^{1/3}.$$

Then for all integers j,

$$\sum_{\substack{1 \le n \le x \\ \Omega(n) \equiv j \pmod k}} 1 = \frac{x}{k} + O(x \exp\{-(\text{loglog } x)^{1/3}\}).$$

Proof: Let z be a complex number with $|z| = 1$. Then it follows from a result due to Selberg [6] that

$$S_z(x) = \sum_{1 \le n \le x} z^{\Omega(n)} = \frac{A(z)x}{(\log x)^{1-z}} + O\left(\frac{1}{|(\log x)^{2-z}|}\right), \tag{1}$$

where $A(z)$ is analytic for $|z| < 2$. Note that

$$\sum_{\substack{1 \le n \le x \\ \Omega(n) \equiv j \pmod k}} 1 - \frac{S_1(x)}{k} = \sum_{1 \le n \le x} \frac{1}{k} \sum_{\ell=1}^{k-1} e^{2\pi i(\Omega(n) - j)\ell/k}$$

$$= \frac{1}{k} \sum_{\ell=1}^{k-1} e^{-2\pi i j\ell/k} \cdot S_{\rho^\ell}(x), \tag{2}$$

where

$$\rho = \exp\{2\pi i/k\}.$$

From (1) we deduce that the largest term on the right of (2) arises out of the root of unity with largest real part. Since $S_1(x) = [x]$, the largest integer $\le x$, we get from (2)

$$\sum_{\substack{1 \le n \le x \\ \Omega(n) \equiv j \pmod k}} 1 = \frac{x}{k} + O(x(\log x)^{\cos(2\pi/k) - 1}). \tag{3}$$

Lemma 1 follows from (3) with a little computation.

Lemma 2: Let $x > 20$ and k be a positive integer satisfying

$$k \le \frac{3}{2} \text{ loglog } x.$$

Then for all integers j, we have

$$\sum_{\substack{1 \le n \le x \\ \Omega(n) \equiv j \pmod{k}}} 1 \ll \frac{x}{k^{1/3}}.$$

Proof: We may assume that $j \ge 1$. We rewrite the sum in the lemma as

$$\sum_{\substack{1 \le n \le x \\ \Omega(n) \equiv j \pmod{k}}} 1 = \sum_{\substack{\nu \equiv j \pmod{k} \\ 0 < \nu \le (3/2)\log\log x}} \sum_{\substack{1 \le n \le x \\ \Omega(n) = \nu}} 1 + \theta \sum_{\substack{1 \le n \le x \\ \Omega(n) > (3/2)\log\log x}} 1, \tag{4}$$

where $0 \le \theta \le 1$.

To estimate the first term on the right of (4) we use the following result due to Sathe and Selberg (see [6]):

$$\sum_{1 \le n \le x,\ \Omega(n) = \nu} 1 = O\left(\frac{x(\log\log x)^{\nu - 1}}{\log x \cdot (\nu - 1)!}\right) \text{ for } 1 \le \nu \le \frac{3}{2}\log\log x.$$

So, the term is

$$\ll \frac{x}{\log x} \sum_{\substack{m = 0 \\ m \equiv j \pmod{k}}}^{\infty} \frac{(\log\log x)^m}{m!}. \tag{5}$$

Set $y = \log\log x$. So the sum in (5) is

$$\frac{x}{\log x} \cdot \frac{1}{k} \sum_{w^k = 1} \frac{e^{wy}}{w^{j-1}} \ll \frac{x}{\log x} \cdot \frac{1}{k} \sum_{w^k = 1} e^{uy}, \tag{6}$$

where w ranges over all kth roots of unity and $w = u + iv$.

First, we assume that

$$\{\log\log x / \log\log\log x\}^{1/3} \le k \le \frac{3}{2}\log\log x.$$

Then

$$\sum_{w^k = 1} e^{uy} = \sum_{\ell = 0}^{k-1} \exp\left\{\left(\cos\frac{2\pi\ell}{k}\right)y\right\} = \sum_{\substack{\ell \le k^{2/3} \text{ or} \\ k - k^{2/3} \le \ell \le k}} + \sum_{k^{2/3} < \ell < k - k^{2/3}} = S_1 + S_2. \tag{7}$$

Clearly

$$S_1 \le 2k^{2/3}e^y \ll k^{2/3} \cdot \log x. \tag{8}$$

To estimate S_2, write

$$\exp\left\{\left(\cos\frac{2\pi\ell}{k}\right)y\right\} = e^y \exp\left\{\left(\left(\cos\frac{2\pi\ell}{k}\right) - 1\right)y\right\} \tag{9}$$

and observe that

$$-\left\{\left(\cos\frac{2\pi\ell}{k}\right) - 1\right\} \gg \frac{1}{k^{2/3}}. \tag{10}$$

From (7), (9), and (10) we deduce that

$$S_2 \ll k^{2/3} \log x. \tag{11}$$

If we combine (6), (7), (8), and (11), we see that the first term on the right of (4) is $\ll x/k^{1/3}$ if

$$\{\log\log x / \log\log\log x\}^{1/3} \le k \le \frac{3}{2}\log\log x. \tag{12}$$

The last term in (4) is easily bounded by appealing to the following theorem of Turán (see [4, pp. 356-358]):

$$\sum_{1 \le n \le x} \{\Omega(n) - \log\log x\}^2 \ll x \log\log x. \tag{13}$$

That is, if $N(x)$ is the number of $n \le x$ for which

$$\Omega(n) > \frac{3}{2} \log\log x,$$

then (13) shows that

$$N(x) \cdot (\log\log x)^2 \ll x \log\log x,$$

whence

$$N(x) \ll x/\log\log x \ll x/k^{1/3}. \tag{14}$$

Thus, we have established Lemma 2, for k satisfying (12). On the other hand, if $k \le \{\log\log x/\log\log\log x\}^{1/3}$, then Lemma 2 follows from Lemma 1.

Proof of the Theorem: For $\eta > 0$, define

$$k(n, \eta) = \prod_{p \le \eta,\ p|(n, \Omega(n))} p \quad \text{and} \quad N_\eta = \prod_{p \le \eta} p.$$

Then

$$Q(x) = \sum_{\substack{1 \le n \le x \\ k(n, \eta) = 1}} 1 + \theta' \sum_{\substack{1 \le n \le x \\ \exists p > \eta,\ p|(n, \Omega(n))}} 1 \tag{15}$$

$$= S_3 + S_4, \text{ respectively,}$$

where $-1 \le \theta' \le 0$. But

$$\begin{aligned} S_3 = \sum_{1 \le n \le x} \sum_{d|k(n,\eta)} \mu(d) &= \sum_{d|N_\eta} \mu(d) \sum_{\substack{1 \le n \le x \\ d|(n, \Omega(n))}} 1 \\ &= \sum_{d|N_\eta} \mu(d) \sum_{\substack{1 \le m \le x/d \\ \Omega(m) \equiv -\Omega(d) \pmod d}} 1 . \end{aligned} \tag{16}$$

In (15) we will choose η such that the integers d in (16) satisfy

$$d \le \left\{\left(\log\log \frac{x}{d}\right) \Big/ \log\log\log\left(\frac{x}{d}\right)\right\}^{1/3}.$$

The Prime Number Theorem (see [4, p. 9]) shows that

$$\eta = \frac{1}{4} \log\log\log x \tag{17}$$

is a permissible choice.

With this choice of η in (16), Lemma 1 shows that

$$\begin{aligned} S_3 &= \sum_{d|N_\eta} \mu(d) \left\{\frac{x}{d^2} + O\left(\frac{x}{d \log\log x}\right)\right\} \\ &= x \sum_{d|N_\eta} \frac{\mu(d)}{d^2} + O\left(\frac{x}{\log\log x} \sum_{d|N_\eta} \frac{1}{d}\right). \end{aligned} \tag{18}$$

First

$$\sum_{d|N_\eta} \frac{\mu(d)}{d^2} = \sum_{d \le \eta} \frac{\mu(d)}{d^2} + O\left(\sum_{d < \eta} \frac{1}{d^2}\right) = \frac{6}{\pi^2} + O\left(\frac{1}{\eta}\right). \tag{19}$$

Also

$$\sum_{d|N_\eta} \frac{1}{d} = \prod_{p \leq \eta}\left(1 + \frac{1}{p}\right) \ll \log \eta. \tag{20}$$

From (17), (18), (19), and (20), we see that

$$S_3 = \frac{6x}{\pi^2} + O\left(\frac{x}{\eta}\right). \tag{21}$$

To estimate S_4, we note that

$$\begin{aligned} |S_4| \leq & \sum_{\substack{1 \leq n \leq x \\ \exists p > \eta,\ p \leq (3/2)\log\log x \\ p|(n,\ \Omega(n))}} 1 + \theta'' \sum_{\substack{1 \leq n \leq x \\ \exists p > (3/2)\log\log x \\ p|(n,\ \Omega(n))}} 1 \\ & = S_5 + S_6, \text{ respectively,} \end{aligned} \tag{22}$$

where $0 \leq \theta'' \leq 1$. Lemma 2 shows that

$$S_5 \leq \sum_{\eta < p \leq (3/2)\log\log x} \sum_{\substack{m \leq x/p \\ \Omega(m) \equiv -1 \pmod p}} 1 \ll \sum_{p > \eta} \frac{x}{p^{4/3}}. \tag{23}$$

From the Prime Number Theorem and (23), we deduce that

$$S_5 \ll \frac{x}{\eta^{1/3} \log \eta}. \tag{24}$$

With regard to S_6, note that

$$S_6 \leq \sum_{\substack{1 \leq n \leq x \\ \Omega(n) > (3/2)\log\log x}} 1 \ll \frac{x}{\log\log x} \tag{25}$$

by the use of (14).

Finally, by combining (15), (21), (22), (24), and (25), we arrive at

$$Q(x) = \frac{6x}{\pi^2} + O\left(\frac{x}{\eta^{1/3} \log \eta}\right). \tag{26}$$

The theorem follows from (26) and (17).

Remarks: With a little more care, our theorem can be improved to

$$Q(x) = \frac{6x}{\pi^2} + O_\varepsilon(x(\log\log\log x)^{-1/2+\varepsilon}),$$

where $\varepsilon > 0$ is arbitrarily small.

If $n > 0$ is a randomly chosen square-free integer, and m a randomly chosen positive integer, then $(n, m) = 1$ with probability

$$c = \prod_p \left(1 - \frac{1}{p^2 + p}\right).$$

By suitably modifying the proof of our theorem, we can show that if n is square free, then $(n, \Omega(n)) = 1$ with probability c. Thus, $\Omega(n)$ behaves randomly with respect to n, even in the square-free case.

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PROPORTIONAL ALLOCATION IN INTEGERS

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Dedicated to the memory of Vern Hoggatt

The U.S. Constitution mandates that "Representatives shall be apportioned among the several states according to their respective numbers. . . . The number of representatives shall not exceed one for every thirty thousand, but each state shall have at least one representative." Implementation is left to Congress.

Controversy arose over the first reapportionment. Congress passed a bill based on a method supported by Alexander Hamilton. President George Washington used his first veto to quash this bill, and an apportionment using Thomas Jefferson's method of "greatest divisors" was adopted. This matter is still controversial. Analyses, reviews of the history, and proposed solutions are contained in the papers [3], [4], and [5] in the *American Mathematical Monthly*.

The purpose of this paper is to cast new light on various methods of proportional allocation in natural numbers by moving away from the application to reapportionment of the House of Representatives after a census and instead considering the application to division of delegate positions among presidential candidates based on a primary in some district.

1. THE MATHEMATICAL PROBLEM

Let $N = \{0, 1, 2, \ldots\}$ and let W consist of all vectors $V = (v_1, \ldots, v_n)$ with components v_i in N and dimension $n \geq 2$. Let the *size* of such a V be

$$|V| = v_1 + \cdots + v_n.$$

An allocation method is a function F from $N \times W$ into W such that

$$F(s, V) = S = (s_1, \ldots, s_n) \text{ with } |S| = s.$$

We will sometimes also write $F(s, V)$ as $F(s; v_1, \ldots, v_n)$. $S = F(s, V)$ should be the vector in W with size s and the same dimension as V which in some sense is most nearly proportional to V.

A property common to all methods discussed below is the *fairness* property that

$$s_i \geq s_j \text{ whenever } v_i > v_j. \tag{1}$$

Note that $s_i > s_j$ can occur with $v_i = v_j$ since the requirement that each s_i be an integer may necessitate use of tie-breaking (e.g., when all v_i are equal and s/n is not an integer).