Note that, by (4.4) and (4.5), (7.6) holds for all nonnegative p. Substituting from (7.6) in (6.1) and (6.2) and evaluating coefficients of x^m, we obtain the following two identities.

$$\begin{aligned}(p+q)T_{r,m}^{(p+q)} &= pT_{r,m}^{(p)} + qT_{r,m}^{(q)} + pq\sum_{s=0}^{r-1}\sum_{j=0}^{m} T_{s,j}^{(p)}T_{r-s-1,m-j}^{(q)} \\ &\quad - pq\sum_{s=0}^{r-1}\sum_{j=0}^{m-1} T_{s,j}^{(p)}T_{r-s-1,m-j-1}^{(q)} \qquad (p>0,\ q>0),\end{aligned} \tag{7.7}$$

$$\begin{aligned}(r+1)T_{r,m}^{(p+q)} &= (r+1)T_{r,m}^{(q)} + p\sum_{s=0}^{r-1}\sum_{j=0}^{m}(r-s)T_{s,j}^{(p)}T_{r-s-1,m-j}^{(q)} \\ &\quad - p\sum_{s=0}^{r-1}\sum_{j=0}^{m-1}(r-s)T_{s,j}^{(p)}T_{r-s-1,m-j-1}^{(q)} \qquad (p>0).\end{aligned} \tag{7.8}$$

In particular, for $q = 0$, (7.8) reduces to

$$\begin{aligned}(r+1)T_{r,m}^{(p)} &= \binom{r}{m}^2 + p\sum_{s=0}^{r-1}\sum_{j=0}^{m}\binom{s}{j}^2 T_{r-s-1,m-j}^{(p)} \\ &\quad - p\sum_{s=0}^{r-1}\sum_{j=0}^{m-1}\binom{s}{j}^2 T_{r-s-1,m-j-1}^{(p)} \qquad (p>0).\end{aligned}$$

We remark that (6.1) is implied by (6.2). To see this, multiply both sides of (6.2) by q, interchange p and q, and then add corresponding sides of the two equations. Similarly, it can be verified that (7.3) is implied by (7.4) and (7.7) is implied by (7.8).

REFERENCE

1. E. D. Rainville. *Special Functions*. New York: Macmillan, 1960.

*****

SOME EXTREMAL PROBLEMS ON DIVISIBILITY PROPERTIES OF SEQUENCES OF INTEGERS

PAUL ERDÖS
University of California, Los Angeles CA 90024

Dedicated to the memory of my friend Vern Hoggatt

A sequence of integers $A = \{a_1 < a_2 < \cdots < a_k \le n\}$ is said to have property $P_r(n)$ if no a_i divides the product of r other a's. Property $P(n)$ means that no a_i divides the product of the other a's. A sequence has property $Q(n)$ if the products a_ia_j are all distinct.

Many decades ago I proved the following theorems [2]:

Let A have property P_1 (i.e., no a_i divides any other). Then

$$\max k = \left[\frac{n+1}{2}\right].$$

The proof is easy.

Let A have property P_2 then [$\pi(n)$ is the number of primes not exceeding n]

(1) $$\pi(n) + c_1 n^{2/3}(\log n)^{-2} < \max k \leq \pi(n) + c_2 n^{2/3}(\log n)^{-2}.$$

The c's will denote positive absolute constants not necessarily the same at each occurrence. We will write P_r instead of $P_r(n)$ if there is no danger of confusion.

Probably there is a c for which

(2) $$\max k = \pi(n) + (c + 0(1))n^{2/3}(\log n)^{-2}$$

but I could never prove (2).

Assume next that A has property Q. Then

(3) $$\pi(n) + c_3 n^{3/4}(\log n)^{-3/2} < \max k < \pi(n) + c_4 n^{3/4}(\log n)^{-2/3}.$$

Here too I conjectured

(4) $$\max k = \pi(n) + (c + 0(1))n^{3/4}(\log n)^{-3/2}.$$

I could never prove (4), which seems more difficult than (2).

In this note I consider slightly different problems. Denote by S_n the set of positive integers not exceeding n. Observe that S_n can be decomposed into $1 + \left[\frac{\log n}{\log 2}\right]$ sets having property P_1. To see this, let S consist of the integers $\left[\frac{n}{2^i}\right] < a \leq \left[\frac{n}{2^{i-1}}\right]$. The powers of 2 show that $1 + \left[\frac{\log n}{\log 2}\right]$ is best possible.

Denote by $f_r(n)$ the smallest integer for which S_n can be decomposed as the union of $f_r(n)$ sets having property P_r and $g(n)$ is the smallest integer for which S_n can be decomposed into $g(n)$ sets having property Q. We just observed $f_1(n) = 1 + \left[\frac{\log n}{\log 2}\right]$. We prove

Theorem 1:

(5) $$c\,\frac{n^{1/2}}{\log n} < f_2(n) < 2n^{1/2}.$$

(6) $$c\,\frac{n^{1/3}}{\log n} < g(n) < 2n^{1/2}.$$

The upper bound in (5) and (6) follows immediately from the fact that

$$m \nmid (m + i_1)(m + i_2) \quad \text{if} \quad 1 \leq i_1 \leq i_2 < m^{1/2}.$$

Now we prove the lower bound in (5). The proof will be similar to the proof in [2]. Let S' be the integers of the form

(7) $$pu,\ u < \frac{1}{2}n^{1/2},\ n^{1/2} < p < 2n^{1/2}.$$

Clearly

(8) $$|S_n'| > c\,\frac{n}{\log n}.$$

Now let $a_1 < a_2 < \cdots < a_k$ be a subset of S_n' which satisfies property P_2. We prove that then

(9) $$k < \frac{n^{1/2}}{2} + c\,\frac{n^{1/2}}{\log n} < n^{1/2}.$$

(8) and (9) clearly complete the proof of (5).

Thus we only have to prove (9). Put $a_i = p_i u_i$ where p_i and u_i satisfy (7). Now make correspond to the set $a_1 < \cdots < a_k$ a bipartite graph where the white vertices are the u's and whose black vertices are the primes p_i. To $a_i = p_i u_i$ corresponds the edge joining p_i and u_i. This graph clearly cannot contain a path of length three. To see this, observe that if $a_1 = p_1 u_1$, $a_2 = u_1 p_2$, and $a_3 = p_2 u_2$ is a path of length three then $a_2 | a_1 a_3$, which is impossible. A bipartite graph which contains no path of length three is a forest and hence it is well known and easy to see that the number of its edges is less than the number of its vertices. This proves (9) and completes the proof of (5).

By a more judicious choice of the black and white vertices the lower bound of (5) can be improved considerably. A well known and fairly deep theorem of mine states that the number of integers $m < n$ of the forms $u \cdot v$, where both u and v are not exceeding $n^{1/2}$ is greater than

$$\frac{n}{(\log n)^{\alpha+\varepsilon}}, \quad \alpha = 1 - \frac{1 + \log\log 2}{\log 2}$$

for $n > n_0(\varepsilon)$, and that this choice of α is the best possible [3]. This immediately gives, by our method,

$$f_2(n) > \frac{n^{1/2}}{(\log n)^{\alpha+\varepsilon}}.$$

We do not pursue this further, since we cannot at present decide whether

$$f_2(n) = 0(n^{1/2})$$

is true. The following extremal problem, which I believe is new, is of interest in this connection: Let $1 \le a_1 < \cdots < a_r \le n$ and $1 \le b_1 < \cdots < b_s \le n$ be two sequences of integers. Denote by $1 \le u_1 < \cdots < u_t \le n$ the integers not exceeding n of the form $a_i b_j$. Put

$$h(n) = \max \frac{t}{r + s},$$

where the maximum is extended over all possible choices of the a's and b's. Our proof immediately gives $f_2(n) \ge h(n)$. I can prove

$$h(n) < \frac{n^{1/2}}{(\log n)^{\beta}} \text{ for some } \beta > 0.$$

It would be interesting if it would turn out that for some $\beta < \alpha$,

$$h(n) > \frac{n^{1/2}}{(\log n)^{\beta}}.$$

The upper bound of (6) is obvious, thus to complete the proof of Theorem 1 we only have to prove the lower bound in (6). The proof will again be similar to that of [2]. Let S_n'' be the integers of the form

$$pu < n, \; u < \left|\frac{1}{2}\right| n^{1/3}, \; n^{2/3} < p < 2n^{2/3}. \tag{10}$$

Clearly (by the prime number theorem or a more elementary theorem)

$$|S_n''| > \frac{cn}{\log n}. \tag{11}$$

Now let $a_1 < \cdots < a_k$ be a subset of S_n'' having property Q (i.e., all the products $a_i a_j$ are distinct). We prove

$$k < n^{2/3} + c\frac{n^{2/3}}{\log n}. \tag{12}$$

(11) and (12) clearly give the lower bound of (6); thus to complete the proof of our Theorem we only have to prove (12). Consider a bipartite graph whose

white vertices are the primes $n^{2/3} < p < 2n^{2/3}$ and whose black vertices are the integers

$$u < \frac{1}{2}n^{1/3}.$$

To each $a = pu$, we make correspond the edge joining p to u. This graph cannot contain a C_4, i.e., a circuit of size four. To see this, observe that if p_1, p_2, u_1, and u_2 are the vertices of this C_4 then p_1u_1, p_1u_2, p_2u_1, and p_2u_2 are all members of our sequence and

$$p_1u_1 \cdot p_2u_2 = p_1u_2 \cdot p_2u_1,$$

or the products a_ia_j are not all distinct, which is impossible.

Now let v_i be the valency (or degree) of p_i $(n^{2/3} < p_i < 2n^{2/3})$. We now estimate k, the number of the edges of our graph, as follows: The p_i's with $v_i = 1$ contribute to k at most

$$s < c\,\frac{n^{2/3}}{\log n}.$$

Now let $p_1, \ldots, p_r$ be the primes whose valency v_i is greater than 1. Observe that

$$\sum_{i=1}^{r}\binom{v_i}{2} \le \binom{\left[\frac{1}{2}n^{1/3}\right]}{2} \le \frac{1}{8}n^{2/3} \tag{13}$$

$\left[\frac{1}{2}n^{1/3}\right]$ is the number of u's. If p_i is joined to v_i u's, form the $\binom{v_i}{2}$ pairs of u's joined to p_i. Now, if (13) would not hold, then by the box principle there would be two p's joined to the same two u's, i.e., our graph would contain a C_4, which is impossible. Thus (13) is proved.

From (13) we immediately have

$$\sum_{i=1}^{r} v_i < \frac{n^{2/3}}{\min(v_i - 1)} \le n^{2/3}. \tag{14}$$

(14) clearly implies (12) and hence the proof of our Theorem is complete.

I expect $g(n) < n^{(1/3+\varepsilon)}$ but have not even been able to prove $g(n) = o(n^{1/2})$.

Recall that $f_r(n)$ is the smallest integer for which S_n can be decomposed into $f_r(n)$ sets having property P_r. We have

Theorem 2: For every $\varepsilon > 0$,

$$n^{1-\frac{1}{r}-\varepsilon} < f_r(n) < c_r n^{1-\frac{1}{r}}$$

The proof of Theorem 2 is similar to that of Theorem 1 and will not be given here. Perhaps

$$f_r(n) = o\left(n^{1-\frac{1}{r}}\right).$$

Finally, denote by $F(n)$ the smallest integer for which S_n can be decomposed into $F(n)$ sets $\{A_i\}$, $1 \le i \le F(n)$, having property P.

Using certain results of de Bruijn [1], I can prove that for a certain absolute constant c

$$F(n) = n\exp\left((-c + O(1))(\log n \log\log n)^{1/2}\right). \tag{15}$$

We do not give the proof of (15) here.

Now I discuss some related results and conjectures. Let $a_1 < a_2 < \cdots < a_k$ be the largest subset of S_n for which the sums $a_i + a_j$ are all distinct. Turán and I proved that [4]

$$\max k = (1 + o(1)n^{1/2}$$

and we in fact conjectured

$$\max k = n^{1/2} + O(1). \tag{16}$$

(16) is probably deep, and I offer $500 for a proof or disproof.

I conjectured more than 15 years ago that if $b_1 < \cdots < b_n$ is any sequence of integers then there always is a subsequence

$$b_{i_1} < \cdots < b_{i_s},\ s \geq (1 + o(1))n^{1/2},$$

so that all the sums $b_{i_{j_1}} + b_{i_{j_2}}$ are distinct. Komlós, Sulyok and Szemerédi [5] proved a much more general theorem from which they deduced a slightly weaker form of my conjecture, namely $s > cn^{1/2}$ for some $c < 1$. Denote by $m(n)$ the largest integer so that for every set of n integers $b_1 < \cdots < b_n$ one can find a subsequence of $m(n)$ terms so that the sum of any two terms of the subsequence are distinct. Perhaps $m(n)$ is assumed for S .

Recently I conjectured that if $b_1 < b_2 < \cdots < b_n$ is any sequence of n integers, one can always select a subsequence $b_{i_1} < \cdots < b_{i_s},\ s > (1 + o(1))n^{1/2}$ so that the product of any two b_{i_j}'s is distinct. Straus observed that with $s > cn^{1/2}$ this follows from the Komlós, Sulyok and Szemerédi theorem by a method which he often used. One can change the multiplicative problem to an additive one by taking logarithms and then, by using Hamel bases, one can easily deduce $s > cn^{1/2}$ from the theorem of Komlós, Sulyok and Szemerédi.

Let $1 \leq a_1 < \cdots < a_k \leq n$ be any sequence of k integers, not exceeding n. Denote by $F(k, n)$ the largest integer so that there always is a subsequence of the a's having $F(k, n)$ terms and property P_1. It is easy to see that

$$F(k, n) \geq \frac{k}{1 + \log n} \tag{17}$$

and the powers of 2 show that (17) in general is best possible. It is not difficult to see that if $k \geq cn$ then $F(k, n) \geq g(c)n$ and the best value of $g(c)$ would be easy to determine although I have not done so. It is further easy to see that $g(c)/c \to 0$ if $c \to 0$. If $k < n^{1-\varepsilon}$, then (17) gives the correct order of magnitude except for a constant factor c , and in general the determination of $F(k, n)$ is not difficult.

Many further questions of this type could be asked. For example, denote by $F_2(k, n)$ the largest integer so that our sequence always has a subsequence of $F_2(k, n)$ terms having property P_2. $F_2(k, n)$ seems to be more difficult to handle than $F(k, n)$. It is easy to see that

$$F_2(k, n) > k(2n^{1/2})^{-1},$$

but perhaps this can be improved and quite possibly for every $c > 0$

$$F_2(cn, n)/n^{1/2} \to \infty.$$

The following question seems of some interest to me: Let

$$1 \leq a_1 < \cdots < a_k \leq n.$$

What is the smallest value of k that forces the existence of three (or s) a's, so that the product of every two is a multiple of the others? In particular, is it true that if $k > cn$ there always are three a's so that the product of any two is a multiple of the third? At the moment I cannot answer this question, but perhaps I overlooked a trivial argument.

To end our paper, we state one more question: What is the smallest $k = k_n$ for which $F_2(k, n) \geq 3$? In other words: Determine or estimate the smallest

$k = k_n$ for which for every $1 \le a_1 < \cdots < a_k \le n$ there are three a's, a_{i_1}, a_{i_2}, a_{i_3} so that the product of two is not a multiple of the third. I have no satisfactory answer, but perhaps again I overlooked a trivial argument.

On the other hand, I can get a reasonably satisfactory answer to a slightly modified question.

Theorem 3: Let $1 \le a_1 < \cdots < a_k \le n$ be such that the product of every two a's is a multiple of all the others. Then ($\exp z = e^z$)

$$\max k = \exp\left((1 + o(1))\log 2 \cdot \frac{2}{3} \log n(\log \log n)^{-1}\right). \tag{18}$$

We only outline the proof of Theorem 3. Let $2, 3, \ldots, p_s$ be the primes not exceeding $(1 - \varepsilon)\frac{2}{3} \log n$. Let the a's be the integers of the form

$$u \prod_{i=1}^{s} p_i \tag{19}$$

where u runs through the integers that are the product of $[s/2]$ or fewer of the p's. From the prime number theorem, we easily obtain that all the a's are not exceeding n. To see this, observe that by the prime number theorem

$$\prod_{i=1}^{s} p_i = \exp\left((1 + o(1))(1 - \varepsilon)\frac{2}{3} \log n\right)$$

and

$$u < \left(\prod_{i=1}^{s} p_i\right)^{\frac{1}{2}+o(1)} < \exp\left((1 + o(1))\frac{\log n}{3}\right).$$

Further, by the prime number theorem,

$$s > (1 - \varepsilon)\frac{2}{3} \log n(\log \log n)^{-1},$$

and the number of u's is not less than 2^{s-1}, which proves the lower bound in (18).

Now we outline the proof of the upper bound of (18). Let $p_1, \ldots, p_s$ be the prime factors of

$$\prod_{i=1}^{k} a_i.$$

Since $a_i a_j$ is a multiple of all the other a's, all but one of the a's, say $a^{(j)}$, are multiples of p_j, $1 \le j \le s$. Disregarding these $a^{(j)}$'s, we assume that all the a's are multiples of all the p_j's. By the same argument we can assume that for every p_j there is an α_j so that every a_i divides p_j with an exponent $x_{i,j}$, $\alpha_j \le x_{i,j} \le 2\alpha_j$. From this and the prime number theorem we obtain by a simple computation, the details of which I suppress, the upper bound in (18).

REFERENCES

1. N. G. de Bruijn. "On the Number of Positive Integers $< x$ and Free of Prime Factors $> y$." *Indagationes Math.* 13 (1951):50-60; see also "On the Number of Uncancelled Elements in the Sieve of Eratosthenes." *Ibid.* 12 (1950): 247-256.
2. P. Erdös. "On Sequences of Integers No One of Which Divides the Product of Two Others and on Some Related Problems." *Izv. Nauk. Inst. Math. Mech. Tomsk* 2 (1938):74-82; see also "On Some Applications of Graph Theory to Number-Theoretic Problems." *Publ. Ramanujan Institute* 1 (1969):131-136.
3. P. Erdös. "An Asymptotic Inequality in the Theory of Numbers." *Vestnik Leningrad Univ.* 15, No. 13 (1960):41-49 (Russian with an English summary).
4. P. Erdös & P. Turán. "On a Problem of Sidon in Additive Number Theory and on Some Related Problems." *J. London Math. Soc.* 16 (1941):212-216.

*****