AN IMPLICIT TRIANGLE OF NUMBERS

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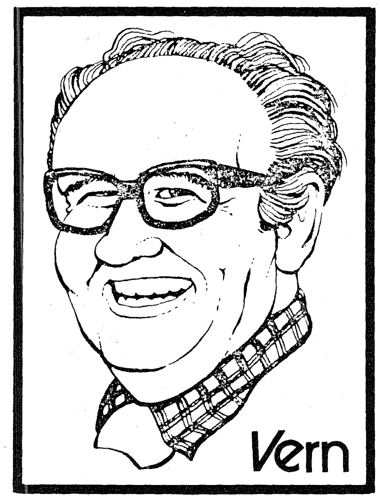

Sketch reprinted from *California MathematiCs* 5(2), October, 1980.

To Vern Hoggatt, whose common sense, plain language, and energetic enthusiasm brought real mathematics into the lives of diverse people throughout the world.

This elementary note introduces a new triangle of numbers that is implicitly defined in Pascal's Triangle. It shares many properties with Pascal's Triangle, including the generation of Fibonacci numbers. It differs from Pascal's Triangle in that it is not symmetrical (and therefore is not a special case of the Fontené-Ward Triangle [1]). When I asked Vern Hoggatt—who seemed to know everything there is to know about Pascal's Triangle—about the Implicit Triangle, he surprised me by replying that he did not know of either the triangle or any of its properties. Therefore, the following may add to our readers' list of "Neat Little Facts about Integers."

The question that led to the discovery of the Implicit Triangle is: "How do we get the squares out of Pascal's Triangle?" One fairly well-known way is to note that

$$0 + 1 = 1,\ 1 + 3 = 4,\ 3 + 6 = 9,\ \ldots,\ \binom{n+1}{2} + \binom{n}{2} = n^2.$$

This can be generalized using Eulerian numbers, so that

$$0 + 4(0) + 1 = 1,\ 0 + 4(1) + 4 = 8,\ 1 + 4(4) + 10 = 27,\ \ldots,$$

$$\binom{n}{3} + 4\binom{n+1}{3} + \binom{n+2}{3} = n\ ,\ \binom{n}{4} + 11\binom{n+1}{4} + 11\binom{n+2}{4} + \binom{n+3}{4} = n^4,$$

etc. See [2], for example. But there is another way to get the squares out of Pascal's Triangle, and this is not so well known:

$$\binom{0}{1} + \binom{0}{2} + \binom{1}{1} + \binom{1}{2} = 1,\ \binom{1}{1} + \binom{1}{2} + \binom{2}{1} + \binom{2}{2} = 4,$$

$$\binom{2}{1} + \binom{2}{2} + \binom{3}{1} + \binom{3}{2} = 9,\ \ldots,\ \binom{n-1}{1} + \binom{n-1}{2} + \binom{n}{1} + \binom{n}{2} = n^2.$$

These squares are generated by adding rhombuses of entries from Pascal's Triangle. By adding other rhombuses, we generate our new triangle:

```
                    0—1— -1 —1
                   / 2 / 1 / 0 /
                  0—1—0—0—0
                 / 2 / 3 / 1 / 0 /
                0—1—1—0—0—0
               / 2 / 5 / 4 / 1 / 0 /
              0—1—2—1—0—0—0
             / 2 / 7 / 9 / 5 / 1 / 0 /
            0—1—3—3—1—0—0—0
           / 2 / 9 /16 /14 / 6 / 1 / 0 /
          0—1—4—6—4—1—0—0—0
         / 2 /11 /25 /30 /20 / 7 / 1 / 0 /
        0—1—5—10—10—5—1—0—0—0
       / 1 /13 /36 /55 /55 /27 / 8 / 1 / 0 /
      0—1—6—15—20—15—6—1—0—0—0
     / 2 /15 /49 /91 /105/ 77/ 35/ 9 / 1 / 0 /
    0—1—7—21—35—35—21—7—1—0—0—0
   / 2 /17 /64 /140/196/182/112/ 44/10 / 1 / 0 /
  0—1—8—28—56—70—56—28—8—1—0—0—0
 / 2 /19 /81 /204/336/378/294/156/ 54/11 / 1 / 0 /
0—1—9—36—84—126—126—84—36—9—1—0—0   0
```

Suppressing the entries from Pascal's Triangle, we get the (almost) triangular array:

```
                    2   1
                  2   3   1
                2   5   4   1
              2   7   9   5   1
            2   9  16  14   6   1
          2  11  25  30  20   7   1
        2  13  36  55  50  27   8   1
      2  15  49  91 105  77  35   9   1
    2  17  64 140 196 182 112  44  10   1
  2  19  81 204 336 378 294 156  54  11   1
```

This Implicit Triangle has the generating formula

$$I(n,\ k) = \binom{n-2}{k-1} + \binom{n-2}{k} + \binom{n-1}{k-1} + \binom{n-1}{k},$$

where $I(n,\ k)$ is the Implicit Triangle entry in the nth row, kth diagonal,

$$n = 1,\ 2,\ 3,\ \ldots,\ k = 0,\ 1,\ 2,\ 3,\ \ldots\ .$$

(The zeroth row is missing from this new triangle.)

Although it lacks the symmetry of Pascal's Triangle, the Implicit Triangle shares many of its properties.

Theorem 1: $I(n-1, k-1) + I(n-1, k) = I(n, k)$.

Proof: This version of Pascal's Identity follows from that identity in Pascal's Triangle.

$$\begin{aligned} I(n-1, k-1) + I(n-1, k) &= \binom{n-3}{k-2} + \binom{n-3}{k-1} + \binom{n-2}{k-2} + \binom{n-2}{k-1} \\ &\quad + \binom{n-3}{k-1} + \binom{n-3}{k} + \binom{n-2}{k-1} + \binom{n-2}{k} \\ &= \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k-1} + 2\binom{n-2}{k} \\ &= \binom{n-2}{k-1} + \binom{n-2}{k} + \binom{n-1}{k-1} + \binom{n-1}{k} \\ &= I(n, k). \end{aligned}$$

This is not really surprising, since the Implicit entries are linear combinations of Pascal entries, and these linear combinations carry along the properties of Pascal's Triangle.

Theorem 2 ("Christmas Stocking Theorem"):

$$\sum_{n=k}^{k+r} I(n, k) = I(k + r + 1, k + 1).$$

Theorem 3 ("Hockey Stick Theorem"):

$$I(n, r) = \sum_{k=r+1}^{k=n} (-1)^{k-r-1} I(n + 1, k).$$

Theorem 4 ("Fibonacci Number Theorem"):

$$\sum_{k=0}^{\infty} I(n - k, k) = F_{n+2}.$$

(∞ exploits the fact that proceeding up a diagonal we eventually get all 0's.)

Theorem 5 ("Alternating Row Sum Theorem"):

$$\sum_{k=0}^{n+1} (-1)^k I(n, k) = 0, \quad n = 2, 3, 4, \ldots .$$

Proofs: All of these theorems follow from the fact that the Implicit entries are linear combinations of the Pascal entries.

And then there are properties different from, but analogous to, properties of Pascal's Triangle. For examples,

Theorem 6 ("Lucas Number Theorem"):

$$\sum_{k=0}^{\infty} I(n - k, n + 1 - 2k) = L_{k+1}.$$

Theorem 7 ("Row Sum Theorem"):

$$\sum_{k=0}^{n} I(n, k) = 2^{n-1}(3).$$

Proofs: Both of these theorems may be proved just as their analogues are proved for Pascal's Triangle. Theorem 7 may be proved very easily with the aid of Theorem 8.

Theorem 8 ("Coefficient Theorem"): $I(n, k)$ is the coefficient of x^{n-k} in the expansion of $(2x + 1)(x + 1)^{n-1}$.

Proof: From the identity

$$I(n, k) = \binom{n-1}{k-1} + 2\binom{n-1}{k},$$

we can see that the Implicit Triangle is formed from the binomial coefficients of two overlapping Pascal Triangles:

$$I(n, k) = \binom{n}{k} + \binom{n-1}{k}.$$

The theorem then follows from the fact that

$$(2x + 1)(x + 1)^{n-1} = x(x + 1)^{n-1} + (x + 1)^n.$$

We are now in a position to look at a Generalized Implicit Triangle:

$$\begin{array}{ccccccccccccc}
 & & & & & a & & 1 & & & & & \\
 & & & & a & & (a+1) & & 1 & & & & \\
 & & & a & & (2a+1) & & (a+2) & & 1 & & & \\
 & & a & & (3a+1) & & (3a+3) & & (a+3) & & 1 & & \\
 & a & & (4a+1) & & (6a+4) & & (4a+6) & & (a+4) & & 1 & \\
a & & (5a+1) & & (10a+5) & & (10a+10) & & (5a+10) & & (a+5) & & 1 \\
\cdots & & & & & & \cdots & & & & & & \cdots
\end{array}$$

Here the generating identity is

$$G(n, k) = G(n - 1, k - 1) + G(n - 1, k), \; G(n, 0) = a, \; G(n, n) = 1;$$

for $a = 1$, this is just Pascal's Identity.

Theorem 9 ("Generalized Coefficient Theorem"): $G(n, k)$ is the coefficient of x^{n-k} in the expansion of $(ax + 1)(x + 1)^{n-1}$.

Proof: The Generalized Implicit Triangle is again just the overlap of Pascal's Triangle and Pascal's Triangle with every entry multiplied by $a - 1$. The theorem follows from the identity

$$(ax + 1)(x + 1)^{n-1} = (x + 1)^n + (a - 1)x(x + 1)^{n-1}.$$

Since each entry of the Generalized Implicit Triangle is a linear combination of entries from Pascal's Triangle, those foregoing theorems whose proofs were based on linear combinations will hold in the general case, with appropriate modifications; for example, the row sums will be of the form $2^{n-1}(a + 1)$.

Had Vern Hoggatt been able to coauthor this article he would no doubt have found many more results. Perhaps our readers will celebrate his memory by looking for further results themselves.

REFERENCES

1. H. W. Gould. "The Bracket Function and Fontené-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients." *The Fibonacci Quarterly* 7 (1969):23-40.
2. Dave Logothetti. "Rediscovering the Eulerian Triangle." *California Mathematics* 4 (1979):27-33.

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FRACTIONAL PARTS $(nr - s)$, ALMOST ARITHMETIC SEQUENCES, AND FIBONACCI NUMBERS

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To the memory of Vern Hoggatt, with gratitude and admiration.

Except where noted otherwise, sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are understood to satisfy the following requirements, as stated for $\{a_n\}$:

(i) the indexing set $\{n\}$ is the set of *all* integers;
(ii) a_n is an integer for every n;
(iii) $\{a_n\}$ is a strictly increasing sequence;
(iv) the least positive term of $\{a_n\}$ is a_1.

We call $\{a_n\}$ *almost arithmetic* if there exist real numbers u and B such that

$$|a_n - un| < B \tag{1}$$

for all n, and we write $a_n \sim un$ if (1) holds for some B and all n.

Suppose r is any irrational number and s is any real number. Put

$$c_m = [mr - s] = \text{the greatest integer less than or equal to } mr - s,$$

and let b be any nonzero integer. It is easy to check that $c_{m+b} - c_m = [br]$, if $(mr - s) < (-br)$, and $= [br] + 1$, otherwise.

Let a_n be the nth term of the sequence of all m satisfying $c_{m+b} - c_m = [br]$.

In the following examples, $r = (1 + \sqrt{5})/2$, the golden mean, and $s = 1/2$. Selected values of m and c_m are:

-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
-9, -7, -6, -4, -3,-1, 1, 2, 4, 5, 7, 9,10,12,14, 15, 17, 18, 20, 22, 23, 25.

When $b = 1$ we have $[br] = 1$, and selected values of n and a_n are:

-1, 0, 1, 2, 3, 4, 5, 6
-4, -2, 1, 3, 6, 9, 11, 14.

When $b = 2$ we have $[br] = 3$, and selected values of n and a_n are:

-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
-5, -4, -3, -2, 0, $\underline{1}$, $\underline{2}$, $\underline{3}$, $\underline{5}$, 6, $\underline{8}$, 9, 10, 11, $\underline{13}$, 14.

Note here the presence of Fibonacci numbers among the a_n. Methods given in this note can be used to confirm that the Fibonacci sequence is a subsequence of $\{a_n\}$ in the present case.