ON THE "$QX + 1$ PROBLEM," Q ODD

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Dedicated to the memory of V. E. Hoggatt

INTRODUCTION

In a previous paper [3] we studied the function

$$f(n) = \begin{cases} (3n + 1)/2 & n \text{ odd} > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

and showed that there are no nontrivial *circuits* for this function which are cycles. In the present paper we shall consider the analogous problem for

$$h(n) = \begin{cases} (qn + 1)/2 & n \text{ odd} > 1,\ q \text{ odd} \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

for $q = 5$ and 7 and shall find all circuits which are cycles for these functions.

I. THE CASE $Q = 5$

Let

$$f(n) = \begin{cases} (5n + 1)/2 & n \text{ odd} > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

Theorem 1: Let $v_2(m)$ be the highest power of 2 dividing m, $m \in Z$ and let n be an odd integer > 1. Then, $n < f(n) < \cdots < f^k(n)$, and $f^{k+1}(n) < f^k(n)$, where $k = v_2(3n + 1)$.

Proof: Let $n_0 = n_1$, $n = f^i(n)$, $i > 1$. Suppose $n_1, n_2, \ldots, n_{j-1}$ are all odd. Then

$$\begin{aligned} 5n_0 + 1 &= 2n_1 \\ 5n_1 + 1 &= 2n_2 \\ &\cdots \\ 5n_{j-1} + 1 &= 2n_j \end{aligned}$$

By simple recursion, we get

$$2^j n_j = 5^j n + \frac{5^j - 2^j}{3}.$$

Thus

$$(1) \qquad 2(3n_j + 1) = 5^j(3n + 1)$$

If $j < v_2(3n + 1)$, then n_j is odd and we may extend the increasing sequence. If $j = v_2(3n + 1)$, then n_j is even. The result follows.

Following [2], let us write $n \xrightarrow{k} m \xrightarrow{\ell} n^*$, where $\ell = v_2(m)$, $n^* = m/2^\ell$, $k = v_2(3m + 1)$ and m is given by $2^k(3m + 1) = 5^k(3n + 1)$.

If we let A be the set of positive odd integers and define $T: A \to A$ by $(Tn) = n^*$, the map from n to n^* is called a *circuit*. Our goal is to prove that

$$13 \xrightarrow{3} 208 \xrightarrow{4} 13 \quad \text{and} \quad 1 \xrightarrow{2} 8 \xrightarrow{3} 1$$

are the only circuits which are cycles under F. We shall accomplish this by reducing our problem to a Diophantine equation and then using a result of Baker [1] to solve the equation.

Theorem 2: There exists n such that $T(n) = n$ only if there are positive integers k, ℓ, and h satisfying

$$(2) \qquad (2^{k+\ell} - 5^k)h = 2^\ell - 1.$$

Proof: Suppose $T(n) = n$. Then $2^k(3m + 1) = 5^k(3n + 1)$ and $2^\ell n = m$. If we write

$$3m + 1 = 5^k h$$
$$3n + 1 = 2^k h$$

We get

$$(2^{k+1} - 5^k)h = 2^\ell - 1$$

as required.

We note that the converse of this theorem is *false*: $k = 1$, $\ell = 2$, $h = 1$ yields a solution of (2), but this solution does not yield integer values of m and n.

Theorem 3: The only solutions of equation (2) in positive integers are

$$k = 1,\ \ell = 2,\ h = 1$$
$$k = 2,\ \ell = 3,\ h = 1$$
$$k = 3,\ \ell = 2,\ h = 1.$$

Proof: We reduce (2) to an inequality in the linear forms of algebraic numbers and then apply the following theorem of Baker [1, p. 45]: Theorem 3. If $\alpha_1, \ldots, \alpha_n$, $n \geq 2$, are nonzero algebraic numbers with degrees and heights at most $d(\geq 4)$ and $A(\geq 4)$ respectively, and if rational integers $b_1, \ldots, b_n$ exist with absolute values at most B such that

$$0 < |b_1 \ln \alpha_1 + \cdots + b_n \ln \alpha_n| < -e^{-\delta B},$$

where $0 < \delta < 1$ and the logarithms have their principal values, then

$$B < (4^{n^2}\delta^{-1}d^{2n} \log A)^{(2n+1)^2}.$$

Returning to (2) we find that the only solutions for $k < 4$ are

$$k = 1,\ \ell = 2,\ h = 1$$
$$k = 2,\ \ell = 3,\ h = 1$$
$$k = 3,\ \ell = 4,\ h = 5.$$

Thus $k \geq 4$ and (2) yields

$$0 < 2^{k+\ell} - 5^k \leq 2^\ell - 1. \tag{3}$$

Dividing both sides of (3) by $2^{k+\ell}$ and using the fact that

$$\frac{1}{2^k - 1} > \ln \frac{2^k}{2^k - 1} \quad \text{for } k \geq 1,$$

we get

$$0 < |(k + \ell) \ln 2 - k \ln 5| < \frac{1}{2^k - 1}, \text{ and hence} \tag{4}$$

$$0 < \left|\frac{\ell}{k} - \log_2 \frac{5}{2}\right| < \frac{1}{k \ln 2(2^k - 1)}$$

Since $\ln 2(2^k - 1) > 2k$ for $k \geq 4$, we see that if $k \geq 4$, ℓ/k must be a convergent in the continued fraction expansion of $\log_2(5/2)$. With the aid of a computer, we find that the first 7 convergents of this continued fraction are

$$\frac{1}{1}, \frac{4}{3}, \frac{37}{28}, \frac{78}{59}, \frac{193}{146}, \frac{850}{643}, \text{ and } \frac{5293}{4004},$$

and it is easily verified that if $k > 4$, none of these convergents satisfy (4). Thus we may assume $k > 4004$.

Now we derive a lower bound for the partial quotients in the continued fraction expansion of $\log_2(5/2)$, using the following theorem of Legendre.

Theorem 4: Let θ be a real number, p_n/q_n a convergent in the continued fraction expansion of θ, and a_n the corresponding partial quotient. Then

$$\frac{1}{(a_{n+1} + 2)qn^2} < \left|\theta - \frac{p_n}{q_n}\right|,$$

which yields

$$\frac{1}{(a_{n+1} + 2)k^2} < \left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{k \ln 2(2^k - 1)}$$

Thus, since $k > 4004$,

$$a_{n+1} > \frac{2^k - 1}{k} \ln 2 - 2 > \frac{2^{4004} - 1}{4004} \ln 2 - 2 > 10^{2750}.$$

Thus, any further solution of (4) corresponds to an extremely large partial quotient in the continued fraction expansion of θ. Finally, we derive an upper bound for k. To this end, we note that if $k > 4004$, we have $\ell/k < 1.33$, so

$$\ell + k < 2.33k \quad \text{and} \quad 2^k - 1 > e^{.00233k} > e^{.001(\ell + k)}.$$

Thus (4) becomes

$$0 < |(k + \ell)\ln 2 - k \ln 5| < \frac{1}{2^{k-1}} < e^{-.001B}$$

where $B = \ell + k$.

Now we apply Theorem 3, with $n = 2$, $d = 4$, $A = 4$, $\delta = .001$, and get

$$B = \ell + k < (4^4 \cdot 10^3 \cdot 4^4 \cdot \ln 5)^{25} < (10^{2.41} \cdot 10^3 \cdot 10^{2.41} \cdot 10^{21})^{25} < 10^{201}.$$

Thus $k < 10^{201}$ also. With the aid of a computer and a multiple precision package designed by Ellison, we computed $\log_2(5/2)$ to 1200 decimal places and then computed the continued fraction expansion of $\log_2(5/2)$ until q_n exceeded 10^{201}. The largest partial quotient is found to be 5393, so (2) has no solutions in positive integers other than (1, 2, 1), (2, 3, 1), and (3, 4, 5). Thus the only circuits which are cycles under f are $1 \xrightarrow{2} 8 \xrightarrow{3} 1$ and $13 \xrightarrow{3} 208 \xrightarrow{4} 13$.

[Note: $n = 17$ also gives rise to a cycle under f. But this cycle results from a *double circuit*: $17 \xrightarrow{2} 108 \xrightarrow{2} 27$ and $27 \xrightarrow{1} 68 \xrightarrow{2} 17$.]

It would be of great interest to know if any n other than 17 and 27 gives rise to a cycle under f which is the result of a *multiple circuit*.

II. THE CASE $Q = 7$

For $n \in Z^+$, let

$$g(n) = \begin{cases} \dfrac{7n + 1}{2} & n \text{ odd}, n > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

as in Case I. Then we can prove the following theorem.

Theorem 5: Let $v_2(m)$ be the highest power of 2 dividing m, $m \in Z$ and let n be an odd integer > 1. Then $n < g(n) < \cdots < g^k(n)$ and $g^{k+1}(n) < g^k(n)$, where $k = v_2(5n + 1)$.

Also, the equation corresponding to (1) is

$$2^j(5n_j + 1) = 7^j(5n + 1). \tag{5}$$

Now we write $n \xrightarrow{k} m \xrightarrow{\ell} n^*$ where $\ell = v_2(m)$, $n^* = m/2\ell$, $k = v_2(5n + 1)$, and $2^k(5m + 1) = 7^k(5n + 1)$.

Again, if we let A be the set of positive odd integers and define $T:A \to A$ by $T(n) = n^*$, the map from n to n^* is called a circuit. Our goal is to prove:

Theorem 6: The only positive odd integer n such that $T(n) = n$ is $n = 1$.

Proof: As in Case I, we reduce this problem to solving

$$(2^{k+\ell} - 7)h = 2^\ell - 1 \tag{6}$$

where k, ℓ, and h are positive integers. We shall show that the only solutions of (6) in positive integers are

$$k = 1,\ \ell = 2,\ h = 3 \quad \text{and} \quad k = 2,\ \ell = 4,\ h = 1.$$

First, if $k \leq 4$, we find that the only solutions of (6) are the ones stated in the theorem and that only the first gives rise to a cyclic circuit, namely, $1 \xrightarrow{1} 4 \xrightarrow{2} 1$.

So $k \geq 4$, and as in Case I, we find that

$$0 < 2^{k+\ell} - 7^k \leq 2^\ell - 1, \tag{7}$$

$$0 < |(k + \ell)\ln 2 - k \ln 7| < \frac{1}{2^k - 1} \tag{8}$$

and

$$0 < \left|\frac{\ell}{k} - \log_2 \frac{7}{2}\right| < \frac{1}{k \ln 2(2^k - 1)}. \tag{9}$$

Since $\ln 2(2^k - 1) > 2k$ for $k > 4$, ℓ/k must be a convergent in the continued fraction expansion of $\log_2(7/2)$.

Again, we find that the first 7 convergents of $\log_2(7/2)$ are

$$\frac{1}{1}, \frac{2}{1}, \frac{9}{5}, \frac{47}{26}, \frac{197}{109}, \frac{1032}{571}, \text{ and } \frac{4325}{2393}.$$

Of these, 9/5 does not furnish a solution of (6) and the remainder do not satisfy (9). Thus $k > 2393$. Further, by Theorem 4, we get

$$a_{n+1} > \frac{2^{2393} - 1 \ln 2}{2393} - 2 > 10^{1500}.$$

Finally, we note that if $k > 2393$, we have $\ell/k < 1.81$. So $\ell + k < 2.81k$ and $2^k - 1 > e^{.00281k} > e^{.001(\ell+k)}$. Thus $0 < |(k + \ell)\ln 2 - k \ln 7| < e^{-.001B}$, where $B = \ell + k$. Again, by Theorem 3, with $n = 2$, $d = 4$, $A = 4$, $\delta = .001$,

$$B = \ell + k < (4^4 \cdot 10^3 \cdot 4^4 \ln 7)^{25} < 10^{203}.$$

With the aid of Ellison's package, we computed $\log_2(7/2)$ to 1200 decimal places and then computed the continued fraction expansion of $\log_2(7/2)$ until q exceeded 10^{203}. The largest partial quotient is found to be 197, so (6) has only the solutions stated in the theorem and the only circuit which is a cycle under g is $1 \xrightarrow{1} 4 \xrightarrow{2} 1$.

In a subsequent paper to be published in this journal, we shall study the general case for this problem and present the tables generated during the computation of $\log_2(5/2)$ and $\log_2(7/2)$ for the two cases presented here.

REFERENCES

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