ADVANCED PROBLEMS AND SOLUTIONS

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-330 *Proposed by Verner E. Hoggatt, Jr., San Jose State Univ., San Jose CA*

If θ is a positive irrational number and $1/\theta + 1/\theta^3 = 1$,

$$A_n = [n\theta],\ B_n = [n\theta^3],\ C_n = [n\theta^2],$$

then prove or disprove:

$$A_n + B_n + C_n = C_{B_n}.$$

H-331 *Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon*

For each fixed integer $k \geq 2$, define the k-Fibonacci sequence $\{f_n^{(k)}\}_{n=0}^{\infty}$ by $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leq n \leq k, \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geq k + 1. \end{cases}$$

Letting $\alpha = [(1 + \sqrt{5})/2]$, show:

(a) $f^{(k)} > \alpha^{n-2}$ if $n \geq 3$;

(b) $\{f^{(k)}\}_{n=2}^{\infty}$ has Schnirelmann density 0.

H-332 *Proposed by David Zeitlin, Minneapolis, MN*

Let $\alpha = (1 + \sqrt{5})/2$. Let $[x]$ denote the greatest integer function. Show that after k iterations ($k \geq 1$), we obtain the identity

$$[\alpha^{4p+2}[\alpha^{4p+2}[\alpha^{4p+2}[\ldots]]]] = F_{(2p+1)(2k+1)}/F_{2p+1},\ (p = 0, 1, \ldots).$$

Remarks: The special case $p = 0$ appears as line 1 in Theorem 2, p. 309, in the paper by Hoggatt and Bicknell-Johnson, *The Fibonacci Quarterly* 17(4):306-318. For $k = 2$, the above identity gives

$$[\alpha^{4p+2}[\alpha^{4p+2}]] = F_{5(2p+1)}/F_{2p+1} = L_{4(2p+1)} - L_{2(2p+1)} + 1.$$

SOLUTIONS

Con-Vergent

H-308 *Proposed by Paul S. Bruckman, Corcord, CA*
(Vol. 17, No. 4, Dec., 1979)

Let

$$[a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n} = \frac{p_n(a_1, a_2, \ldots, a_n)}{q_n(a_1, a_2, \ldots, a_n)}$$

denote the nth convergent of the infinite simple continued fraction

$$[a_1, a_2, \ldots], \; n = 1, 2, \ldots .$$

Also, define $p_0 = 1$, $q_0 = 0$. Further, define

$$(1) \qquad \begin{aligned} W_{n,k} &= p_n(a_1, a_2, \ldots, a_n)q_k(a_1, a_2, \ldots, a_k) \\ &\quad - p_k(a_1, a_2, \ldots, a_k)q_n(a_1, a_2, \ldots, a_n) \\ &= p_n q_k - p_k q_n, \; 0 \le k \le n. \end{aligned}$$

Find a general formula for $W_{n,k}$.

Solution by the proposer.

Recall that the p_n's and q_n's satisfy the basic recursion

$$(2) \qquad r_{n+1} = a_{n+1}r_n + r_{n-1}, \; n = 1, 2, \ldots .$$

Also, the following relations are either obvious or well known:

$$(3) \qquad W_{n,n} = 0;$$

$$(4) \qquad W_{n,n-1} = (-1)^n, \; n \ge 1;$$

$$(5) \qquad W_{n,n-2} = (-1)^{n-1}a_n, \; n \ge 2.$$

[See Niven and Zuckerman, *An Introduction to the Theory of Numbers*, 3rd ed. (New York: Wiley, 1972), Theorem 7.5, for a proof of (4) and (5).]

We show, by strong induction, that

$$(6) \qquad W_{n,k} = (-1)^{k+1}p_{n-k-1}(a_{k+2}, a_{k+3}, \ldots, a_n).$$

Let S denote the set of positive integers n such that (6) holds for $0 \le k < n$. Setting $n = 1$ in (4) yields $W_{1,0} = -1 = (-1)^{0+1}p_0$; hence, $1 \in S$. Suppose that for some integer $m \ge 2$, $1, 2, \ldots, m \in S$. By (4) and (5), we have:

$$(7) \qquad W_{m+1,m} = (-1)^{m+1} = (-1)^{m+1}p_0, \text{ and } W_{m+1,m-1} = (-1)^m a_{m+1}, \text{ or}$$

$$(8) \qquad W_{m+1,m-1} = (-1)^{m-1+1}p_1(a_{m+1}).$$

Also, if $0 \le k \le m - 2$,

$$\begin{aligned} W_{m+1,k} &= p_{m+1}q_k - p_k q_{m+1} = (a_{m+1}p_m + p_{m-1})q_k - p_k(a_{m+1}q_m + q_{m-1}) \\ &= a_{m+1}(p_m q_k - p_k q_m) + p_{m-1}q_k - p_k q_{m-1} = a_{m+1}W_{m,k} + W_{m-1,k} \end{aligned}$$

[using (1) and (2)]. Hence, by the inductive hypothesis and (2),

$$\begin{aligned} W_{m+1,k} &= (-1)^{k+1}a_{m+1}p_{m-k-1}(a_{k+2}, \ldots, a_m) + (-1)^{k+1}p_{m-k-2}(a_{k+2}, \ldots, a_{m-1}) \\ &= (-1)^{k+1}p_{m-k}(a_{k+2}, \ldots, a_{m+1}). \end{aligned}$$

Thus, using (7) and (8),

(9) $$W_{m+1,\,k} = (-1)^{k+1}p_{m-k}(a_{k+2}, \ldots, a_{m+1}),\ 0 \le k \le m,$$

which is equivalent to the statement $(m + 1) \varepsilon S$. Hence,

$$1, 2, \ldots, m \varepsilon S \Rightarrow (m + 1) \varepsilon S.$$

By induction, (6) is proved.

Fibonacci and Lucas Are the Greatest Integers

H-310 *Proposed by Verner E. Hoggatt, Jr., San Jose State Univ., San Jose, CA (Vol. 17, No. 4, Dec., 1979)*

Let $\alpha = (1 + \sqrt{5})/2$, $[n\alpha] = a_n$, and $[n\alpha^2] = b_n$. Clearly $a_n + n = b_n$.

(a) Show that if $n = F_{2m+1}$, then $a_n = F_{2m+2}$ and $b_n = F_{2m+3}$.

(b) Show that if $n = F_{2m}$, then $a_n = F_{2m+1} - 1$ and $b_n = F_{2m+2} - 1$.

(c) Show that if $n = L_{2m}$, then $a_n = L_{2m+1}$ and $b_n = L_{2m+2}$.

(d) Show that if $n = L_{2m+1}$, then $a_n = L_{2m+2} - 1$ and $b_n = L_{2m+3} - 1$.

Solution by Paul S. Bruckman, Corcord, CA

We begin by noting that

$$\begin{aligned} F_{n+1} - \alpha F_n &= \frac{1}{\sqrt{5}}\{\alpha^{n+1} - \beta^{n+1} - \alpha(\alpha^n - \beta^n)\} \\ &= \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1} - \alpha^{n+1} - \beta^{n-1}) \\ &= -\beta^n/\sqrt{5}(\beta - \alpha), \end{aligned}$$

or

(1) $$\beta^n = F_{n+1} - \alpha F_n.$$

Also, $\alpha L_n - L_{n+1} = \alpha(\alpha^n + \beta^n) - (\alpha^{n+1} + \beta^{n+1}) = -\beta^n(\beta - \alpha)$, or

(2) $$\beta^n\sqrt{5} = \alpha L_n - L_{n+1}.$$

Since $-1 < \beta < 0$, thus $0 < \beta^{2n} \le 1$ and $-1 < \beta^{2n+1} < 0$ $(n \ge 0)$. Hence, using (1)

$$0 < F_{2n+1} - \alpha F_{2n} \le 1 \quad \text{and} \quad -1 < F_{2n+2} - \alpha F_{2n+1} < 0;$$

note that equality is attained above if and only if $n = 0$. Therefore,

$$F_{2n+1} - 1 \le \alpha F_{2n} < F_{2n+1} \quad \text{and} \quad F_{2n+2} < \alpha F_{2n+1} < F_{2n+2} + 1 \ (n \ge 0).$$

It follows that

(3) $$[\alpha F_{2n}] = F_{2n+1} - 1, \text{ and}$$

(4) $$[\alpha F_{2n+1}] = F_{2n+2} \ (n \ge 0).$$

Now (3) implies $[\alpha^2 F_{2n}] = [(1 + \alpha)F_{2n}] = F_{2n} + [\alpha F_{2n}] = F_{2n} + F_{2n+1} - 1$, or

(5) $$[\alpha^2 F_{2n}] = F_{2n+2} - 1.$$

Also, $[\alpha^2 F_{2n+1}] = F_{2n+1} + [\alpha F_{2n+1}] = F_{2n+1} + F_{2n+2}$, or

(6) $$[\alpha^2 F_{2n+1}] = F_{2n+3}.$$

Note that (4) and (6) are equivalent to (a) of the original problem; also, (3) and (5) are equivalent to (b) of the original problem.

In order to prove (c) and (d), we proceed similarly, using the result in (2). We need only observe that $|\beta^n\sqrt{5}| < 1$ for $n \ge 2$. The desired results then follow, as before, for all values of n except for possibly $n = 0$; however, a quick inspection shows that the results also hold for $n = 0$, i.e.,

(7) $$[\alpha L_{2n}] = L_{2n+1}, \; [\alpha L_{2n+1}] = L_{2n+2} - 1,$$

which imply the other two results.

Comment by *Bob Prielipp, University of Wisconsin-Oshkosh, WI*

Sharp-eyed readers will find that this problem can be solved easily by using the following four lemmas established in the article "Representations of Integers in Terms of Greatest Integer Functions and the Golden Section Ratio" by Hoggatt and Bicknell-Johnson [*The Fibonacci Quarterly* 17(4):306-318].

Lemma 1 (p. 308): $[\alpha F_n] = F_{n+1}$, n odd, $n \geq 2$;

$[\alpha F_n] = F_{n+1} - 1$, n even, $n \geq 2$.

Lemma 2 (p. 308): $[\alpha^2 F_n] = F_{n+2}$, n odd, $n \geq 2$;

$[\alpha^2 F_n] = F_{n+2} - 1$, n even, $n \geq 2$.

Lemma 6 (p. 315): $[\alpha L_n] = L_{n+1}$ for n even, if $n \geq 2$;

$[\alpha L_n] = L_{n+1} - 1$ for n odd, if $n \geq 3$.

Lemma 7 (p. 315): $[\alpha^2 L_n] = L_{n+2}$ if n is even and $n \geq 2$;

$[\alpha^2 L_n] = L_{n+2} - 1$ if n is odd and $n \geq 1$.

Also solved by Bob Prielipp, G. Wulczyn, and the proposers.

CORRECTIONS

1. The problem solved in Vol. 18, No. 2, April 1980 is H-284 not H-285.
2. H-315 as it appeared in Vol. 18, No. 2, April 1980 had several misprints in it. A corrected version is given below.

H-315 *Proposed by D. P. Laurie, National Research Institute for Mathematical Sciences, Pretoria, South Africa*

Let the polynomial P be given by

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

and let $z_1, z_2, \ldots, z_n$ be distinct complex numbers. The following iteration scheme for factorizing P has been suggested by Kerner [1]:

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)}; \; i = 1, 2, \ldots, n.$$

Prove that if $\sum_{j=1}^{n} z_i = -a_{n-1}$, then also $\sum_{i=1}^{n} \hat{z}_i = -a_{n-1}$.

Reference

1. I. Kerner. "Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen." *Numer. Math.* 8 (1966):290-94.

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