ORTHOGONAL LATIN SYSTEMS

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Dedicated to the memory of our friend Vern E. Hoggatt

I. INTRODUCTION

A *Latin square of order n* can be interpreted as a multiplication table for a binary operation on n objects $0, 1, \ldots, n - 1$ with both a right and a left cancellation law. That is, if we denote the operation by $*$, then

$$\begin{aligned} a * b = a * c &\Rightarrow b = c \\ b * a = c * a &\Rightarrow b = c. \end{aligned} \tag{1.1}$$

In a completely analogous manner, a *Latin k-cube of order n* is a k-ary operation on n objects with a cancellation law in every position. That is, for the operation $(\ )_*$,

$$(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_k)_* = (a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_k)_* \tag{1.2}$$

implies $b = c$ for all choices of $i = 1, 2, \ldots, k$ and all choices of

$$\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k\} \subset \{0, 1, \ldots, n - 1\}.$$

We permit 1-cubes which are just permutations of $\{0, 1, \ldots, n - 1\}$.

Two Latin squares are orthogonal if the simultaneous equations

$$x * y = a, \quad x \circ y = b \tag{1.3}$$

have a unique solution x, y for every pair a, b. *A set of Latin squares is orthogonal* if every pair of squares in the set is orthogonal.

In an analogous manner, a *k-tuple of Latin k-cubes is orthogonal* if the simultaneous equations

$$\begin{aligned} (x_1, x_2, \ldots, x_k)_1 &= a_1 \\ (x_1, x_2, \ldots, x_k)_2 &= a_2 \\ &\vdots \\ (x_1, x_2, \ldots, x_k)_k &= a_k \end{aligned} \tag{1.4}$$

have a unique solution $x_1, \ldots, x_k$ for all choices of $a_1, \ldots, a_k$.

A *set of Latin k-cubes is orthogonal* if every k-tuple of the set is orthogonal.

In earlier papers, [1] and [2], we showed that the existence of a pair of orthogonal Latin squares can be used for the construction of a quadruple of orthogonal Latin cubes (3-cubes) and for the construction of orthogonal k-tuples of Latin k-cubes for every $k \geq 3$. In this note, we examine in greater detail what sets of orthogonal Latin k-cubes can be constructed by composition from cubes of lower dimensions.

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II. COMPOSITION OF LATIN CUBES

Let $C = (a_1, \ldots, a_s)$ be a Latin s-cube and let $C_i = (b_{i1}, b_{i2}, \ldots, b_{ik_i})_i$ be Latin k_i-cubes $i = 1, 2, \ldots, s$. Then

$$C^* = (C_1, C_2, \ldots, C_s)$$

is a Latin k-cube, where $k = k_1 + k_2 + \cdots + k_s$.

To see this we need only check that the cancellation law (1.2) holds. Now let all the entries be fixed except for the entry b_{ij} in the jth place of C_i. Since C is a Latin cube it follows that, if the values of C^* are equal for two different entries of b_{ij} then the values of C_i must be equal for those two entries. This contradicts the fact that C_i is a Latin cube.

This composition, while algebraically convenient, is not intuitive and we refer the reader to [1] where we explicitly constructed a quadruple of 3-cubes starting from a pair of orthogonal Latin squares of order 3. In the present notation, starting from $a * b$ and $a \circ b$ as orthogonal Latin squares, we constructed the quadruples

$$(a * b) * c, \quad (a * b) \circ c, \quad (a \circ b) * c, \quad (a \circ b) \circ c$$

or, equivalently,

$$a * (b * c), \quad a * (b \circ c), \quad a \circ (b * c), \quad a \circ (b \circ c)$$

as orthogonal quadruples of cubes.

Similarly, if $(\ )_1, \ldots, (\ )_k$ denote an orthogonal set of Latin k-cubes, then

$$(a_1, \ldots, a_k)_1 \circ a_{k+1}, \quad (a_1, \ldots, a_k)_2 \circ a_{k+1}, \ldots, (a_1, \ldots, a_k)_k \circ a_{k+1},$$
$$(a_1, \ldots, a_k)_i * a_{k+1}$$

is an orthogonal $(k + 1)$-tuple of Latin $(k + 1)$-cubes for any $i \in \{1, \ldots, k\}$. To see this, consider the system of equations

$$(x_1, \ldots, x_k)_j \circ x_{k+1} = a_j, \quad 1 \leq j \leq k$$
$$(x_1, \ldots, x_k)_i * x_{k+1} = a_{k+1}.$$

Then the two simultaneous equations

$$(x_1, \ldots, x_k)_i \circ x_{k+1} = a_i, \quad (x_1, \ldots, x_k)_i * x_{k+1} = a_{k+1}$$

have a unique solution $(x_1, \ldots, x_k)_i$ and x_{k+1}. Once x_{k+1} is determined, the equations

$$(x_1, \ldots, x_k)_j \circ x_{k+1} = a_j$$

determine $(x_1, \ldots, x_k)_j$ for all $j = 1, \ldots, i - 1, i + 1, \ldots, k$. Now by the orthogonality of the k-cubes the values of $x_1, \ldots, x_k$ are determined.

Since pairs of orthogonal Latin squares exist for all orders $n \neq 2, 6$, it follows that there exist orthogonal k-tuples of Latin k-cubes for all k provided the order n is different from 2 or 6. It is obvious that there are no orthogonal k-tuples of Latin k-cubes of order 2 for any $k \geq 2$. For order $n = 6$ and dimension $k > 2$, neither the existence nor the nonexistence of orthogonal k-tuples of k-cubes is known. It is therefore worth mentioning the following conditional fact.

Theorem II-1: If there exists a k-tuple of orthogonal Latin k-cubes of order n then there exists an ℓ-tuple of orthogonal Latin ℓ-cubes of order n for every $\ell = 1 + s(k - 1)$, $s = 0, 1, 2, \ldots$.

Proof: By induction on s. The statement is obvious for $s = 0$. So assume the statement true for ℓ and let $(\ )_1^k, \ldots, (\ )_k^k$ denote the orthogonal k-cubes and let $(\ )_1^\ell, \ldots, (\ )_\ell^\ell$ denote the orthogonal ℓ-cubes. Then we construct the following set of Latin $(\ell + k - 1)$-cubes.

$$(a_1, \ldots, a_{\ell+k-1})_1^{\ell+k-1} = ((a_1, \ldots, a_\ell)_1^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k$$
$$(a_1, \ldots, a_{\ell+k-1})_2^{\ell+k-1} = ((a_1, \ldots, a_\ell)_1^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_2^k$$
$$\vdots$$
$$(a_1, \ldots, a_{\ell+k-1})_k^{\ell+k-1} = ((a_1, \ldots, a_\ell)_1^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_k^k$$
$$(a_1, \ldots, a_{\ell+k-1})_{k+1}^{\ell+k-1} = ((a_1, \ldots, a_\ell)_2^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k$$
$$(a_1, \ldots, a_{\ell+k-1})_{k+2}^{\ell+k-1} = ((a_1, \ldots, a_\ell)_3^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k$$
$$\vdots$$
$$(a_1, \ldots, a_{\ell+k-1})_{\ell+k-1}^{\ell+k-1} = ((a_1, \ldots, a_\ell)_\ell^\ell, a_{\ell+1}, \ldots, a_{\ell+k-1})_1^k.$$

From the orthogonality of $(\ )_1^k, \ldots, (\ )_k^k$ it follows that the equations

$$(x_1, \ldots, x_{\ell+k-1})_i^{\ell+k-1} = a_i; \quad i = 1, \ldots, k$$

determine $(x_1, \ldots, x_\ell)_1^\ell, x_{\ell+1}, \ldots, x_{\ell+k-1}$. Once $x_{\ell+1}, \ldots, x_{\ell+k-1}$ are determined, then the equations

$$(x_1, \ldots, x_{\ell+k-1})_{k+j}^{\ell+k-1} = a_{k+j}; \quad j = 1, \ldots, \ell - 1$$

determine $(x_1, \ldots, x_\ell)_{j+1}^\ell$. Now, by the orthogonality of $(\ )_1^\ell, \ldots, (\ )_\ell^\ell$, this determines $x_1, \ldots, x_\ell$.

III. ORTHOGONAL $(k + 1)$-TUPLES OF LATIN k-CUBES

The above construction yielded a set of 4 orthogonal 3-cubes constructed with the help of a pair or orthogonal Latin squares $a \circ b$ and $a * b$. It is natural to ask whether analogous constructions exist for higher dimensions. At the moment we have only succeeded in doing this for dimensions 4 and 5.

Theorem III-1: The 4-cubes

$$(abcd)_1^4 = (a \circ b) \circ (c \circ d)$$
$$(abcd)_2^4 = (a \circ b) * (c \circ d)$$
$$(abcd)_3^4 = (a * b) \circ (c * d)$$
$$(abcd)_4^4 = (a * b) * (c * d)$$
$$(abcd)_5^4 = (a \circ b) \circ (c * d)$$

form an orthogonal set.

Proof: We need to show that the equations

$$(xyzw)_i^4 = a_i$$

determine x, y, z, w when i runs through any four of the five values. Consider first the case $i = 1, 2, 3, 4$. Then the first two equations determine $x \circ y$, $z \circ w$ and the next two equations determine $x * y$, $z * w$. Now $x \circ y$ and $x * y$ determine x, y and $z \circ w$, $z * w$ determine z, w.

Now assume that one of the first four values of i is omitted. By symmetry we may assume $i \neq 4$. Then the first two equations still determine $x \circ y$, $z \circ w$. Once $x \circ y$ is determined, the last equation determines $z * w$ and once $z * w$ is determined, the third equation determines $x * y$. The rest is as before.

Theorem III-2: Let $(\ )^3_1$, $(\ )^3_2$, $(\ )^3_3$ denote an orthogonal set of 3-cubes. Then the 5-cubes

$$\begin{aligned}(abcde)^5_1 &= (abc)^3_1 \circ (d \circ e)\\ (abcde)^5_2 &= (abc)^3_1 * (d \circ e)\\ (abcde)^5_3 &= (abc)^3_2 \circ (d * e)\\ (abcde)^5_4 &= (abc)^3_2 * (d * e)\\ (abcde)^5_5 &= (abc)^3_3 \circ (d \circ e)\\ (abcde)^5_6 &= (abc)^3_3 \circ (d * e)\end{aligned}$$

form an orthogonal set.

Proof: Consider the set of equations

$$(xyzuv)_i = a_i$$

where i runs through five of the six values. If $i \neq 5$ or 6 then the first two equations determine $(xyz)^3_1$ and $u \circ v$ and the second two equations determine $(xyz)^3_2$ and $u * v$. Thus, u, v are determined and, therefore, the last equation determines $(xyz)^3_3$ and thus x, y, z are determined.

If i omits one of the first four values, we may assume by symmetry $i \neq 4$. Then the first two equations determine $(xyz)^3_1$, and $u \circ v$. Now $i = 5$ determines $(xyz)^3_3$ and thereby $i = 6$ determines $u * v$. Finally, $i = 3$ determines $(xyz)^3_2$, and thus x, y, z, u, v are determined.

Applying these results to the lowest order, $n = 3$, we get the surprising result that there exists a $3 \times 3 \times 3$ cube with 4-digit entries to the base 3, so that each digit runs through the values 0, 1, 2 on every line parallel to an edge of the cube and so that each triple from 000 to 222 occurs exactly once in every position as a subtriple of a quadruple. Similarly, there exists a $3 \times 3 \times 3 \times 3$ cube with 5-digit entries, and all quadruples from 0000 to 2222 occur exactly once in every position as subquadruples of the quintuples. Finally, there exists a $3 \times 3 \times 3 \times 3 \times 3$ cube with 6-digit entries, every digit running through 0, 1, 2 on every line parallel to an edge and every quintuple occurring exactly once in every position as a subquintuple.

There does not appear to exist an obvious extension of Theorems III-1 and III-2 to dimensions greater than 5.

It is possible to use the case $n = 3$ to show that the existence of two orthogonal Latin squares of order n does not imply the existence of more than 4 orthogonal 3-cubes or 5 orghogonal 4-cubes of order n.

Theorem III-3: There do not exist 5 orthogonal 3-cubes of order 3.

Proof: Since relabelling the entries in the cube affects neither Latinity nor orthogonality, we may assume that $(i00)_j = i$ for all the 3-cubes $(\ )_j$. So the entries $(010)_j$ are all 1 or 2. If there are 5 orthogonal 3-cubes, then no 3 of them can have the same entry in the position $(010)_j$, since these triples occur already in the positions $(i00)_j$. But in 5 entries 1 or 2, there must be three equal ones.

Theorem III-4: There do not exist 6 orthogonal 4-cubes of order 3.

Proof: As before, assume $(i00)_j = i$, $j = 1, \ldots, 6$. Since all entries $(010)_j$ are either 1 or 2 and no four of them are equal, we may assume that the entries are 111222 as $j = 1, \ldots, 6$. Hence, the entries $(020)_j$ are 222111 in the same order. Now the entries $(001)_j$ and $(002)_j$ must also be three 1's and three 2's and cannot agree with 111222 or 222111 in four positions. But the agreement is always in an even number of positions, and if the agreement with 111222 is in

$2k$ positions, then the agreement with 222111 is in $6 - 2k$ positions and one of these numbers is at least 4.

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ON THE "QX + 1 PROBLEM," Q ODD—II

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In [1] we studied the functions

$$f(n) = \begin{cases} (5n+1)/2 & n \text{ odd} > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

and

$$g(n) = \begin{cases} (7n+1)/2 & n \text{ odd} > 1 \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

and proved:

1. The only nontrivial circuit of f which is a cycle is

$$13 \xrightarrow{3} 208 \xrightarrow{4} 13.$$

2. The function g has no nontrivial circuits which are cycles.

In this note, we consider briefly the general case for this problem and present the tables generated for the computation of $\log_2(5/2)$ and $\log_2(7/2)$ for the two cases presented in [1].

Let

$$h(n) = \begin{cases} (qn+1)/2 & n \text{ odd}, n > 1, q \text{ odd} \\ n/2 & n \text{ even} \\ 1 & n = 1 \end{cases}$$

Then, as in [1], we have

Theorem 1: Let $v_2(m)$ be the highest power of 2 dividing m, $m \in Z$, and let n be an odd integer > 1, then

$$n < h(n) < \cdots < h^k(n), \text{ and } h^{k+1}(n) < h(n),$$

where $k = v_2((q-2)n + 1)$.

Also, the equation corresponding to Eq. (1) in [1] is

(1) $$2^j((q-2)nj + 1) = q^j((q-2)n + 1).$$

Again, we write

$$n \xrightarrow{k} m \xrightarrow{\ell} n^*$$

where $\ell = v_2(m)$, $n^* = m/2^\ell$, $k = v_2((q-2)n + 1)$ and

$$2^k((q-2)m + 1) = q^k((q-2)n + 1)$$

and obtain our usual definition of a circuit.