$4a$, $5a$ gives three steps more than a, 0, a, $2a$. Hence, we can have many sets of four numbers of the form 0, a, b, c having the same number of steps.

However, we can tell the number of steps of the reduced set 0, a, b, c in the following cases:

0, 0, 0, a ($a > 0$) five rows; 0, 0, a, a ($a > 0$) four rows;
0, 0, a, b ($a < b \leq 2a$) five rows; 0, 0, a, $2a + x$ ($x > 0$) seven rows;
0, 0, a, $na + x$ ($n \geq 3$) seven rows; 0, a, 0, a ($a > 0$) three rows;
0, a, 0, b ($a \neq b$) five rows; 0, a, b, c ($b = a + c$, $a = c > 0$) three rows;
0, a, b, c ($b = a + c$, $a \neq c$) four rows;
0, a, b, c ($c = a + b$, $a = b > 0$) four rows;
0, a, b, c ($c = a + b$, $a < b$) six rows; and
0, a, b, c ($c = a + b$, $a > b$) four rows.

From the above, it is clear that the only case which presents difficulty in deciding the number of steps without actual calculation is

$$0,\ a,\ b,\ c\ (abc \neq 0,\ b \neq a + c,\ c \neq a + b),$$

where we can assume $a < c$.

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ASYMPTOTIC BEHAVIOR OF LINEAR RECURRENCES

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In general, it is difficult to predict at a glance the ultimate behavior of a linear recurrence sequence. For example, in some problems where the sequence represents the value of a physical quantity at various times, we might want to know if the sequence is always positive, or at least positive from some point on.

Consider the two sequences:

$$w_0 = 3,\ w_1 = 3.01,\ w_2 = 3.0201$$

and

$$w_{n+3} = 3.01w_{n+2} - 3.02w_{n+1} + 1.01w_n \quad \text{for } n \geq 0;$$

$$v_0 = 3,\ v_1 = 3.01,\ v_2 = 3.0201$$

and

$$v_{n+3} = 3v_{n+2} - 3.01v_{n+1} + 1.01v_n \quad \text{for } n \geq 0.$$

The sequence $\{w_n\}$ is always positive, but the sequence $\{v_n\}$ is infinitely often positive and infinitely often negative. This last fact is not obvious from looking at the first few terms of $\{v_n\}$ since the first negative term is v_{735}.

Clearly, the behavior of a recurrence sequence depends on the roots of its characteristic polynomial. We will prove some results which make this dependence precise.

Let

(1) $$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_ku_{n-k},\ a_i \in R,\ 1 \le i \le k,$$

denote a kth-order linear recurrence with corresponding characteristic polynomial

$$p(x) = x^k - a_1x^{k-1} - \cdots - a_k.$$

For simplicity we shall assume $p(x)$ has distinct roots (although possibly complex). All the results stated here carry through in the case that $p(x)$ has multiple roots and we invite the interested reader to verify such cases in order to obtain a more complete understanding.

The terms of the sequence $\{u_n\}_{n=0}^{\infty}$ defined by (1) can be expressed in terms of the roots of $p(x)$ by use of the Binet formula as follows:

(2) $$u_n = \sum_{i=1}^{s} c_{r_i} r_i^n + \sum_{i=s+1}^{t} (c_{\alpha_i}\alpha_i^n + \overline{c}_{\alpha_i}\overline{\alpha}_i^n) = \sum_{i=1}^{s} c_{r_i} r_i^n + \sum_{i=s+1}^{t} 2\text{Re}(c_{\alpha_i}\alpha_i^n)$$

where r_i, $1 \le i \le s$, denote the real roots of $p(x)$ and α_i, $s + 1 \le i \le t$ denote the roots with nonzero imaginary parts. It is assumed c_{r_i} and c_{α_i} are nonzero.

We are now ready to determine under what conditions the tail of the sequence $\{u_n\}_{n=0}^{\infty}$ will contain only positive terms. We begin with a definition.

Definition: A sequence $\{u_n\}_{n=0}^{\infty}$ is said to be asymptotically positive (denoted a.p.) if there exists $N \in Z$ such that for all $n \ge N$ we have $u_n > 0$.

We first prove a lemma that will shed light on the effects of a complex root of $p(x)$ on the behavior of the sequence $\{u_n\}_{n=0}^{\infty}$.

Lemma 1: If $\theta \not\equiv 0 \bmod \pi$, then the sequence $\{\cos(\lambda + n\theta)\}_{n=0}^{\infty}$, $\lambda, \theta \in \mathbb{R}$, has infinitely many positive and infinitely many negative terms.

Proof: *Case 1.*—θ is a rational multiple of 2π. Then there exist integers s and t, $(s,t) = 1$, such that $\frac{s}{t}2\pi = \theta$. Since $\theta \not\equiv 0 \bmod \pi$, we have $t \ge 3$. Observing that

$$\cos(\lambda + n\theta) = \text{Re}\{e^{i(\lambda+n\theta)}\},$$

we turn our attention to the points $\{e^{i(\lambda+n\theta)}\}_{n=1}^{t}$ in $\mathbb{C}$. The image points in $\mathbb{C}$ differ in argument by at most $\frac{2}{3}\pi$ radians for any two neighboring points. Thus there is always at least one point in each of the half planes $\text{Re}\{z\} > 0$ and $\text{Re}\{z\} < 0$. Since $\cos(\lambda + n\theta)$ is periodic with period t, and in every t consecutive terms there must be at least one positive and one negative term, the lemma holds.

Case 2.—θ is an irrational multiple of 2π. The sequence $\{\lambda + n\theta\}_{n=0}^{\infty}$ is dense mod 2π. (Indeed, it is uniformly distributed mod 2π [2].) As the cosine is continuous, the image of $\{\lambda + n\theta\}_{n=0}^{\infty}$ under the cosine is dense in $[-1, 1]$. This completes the proof.

We are now ready to state and prove the main result.

Theorem 1: Let u_n be a kth-order linear recurrence as in (1) whose characteristic polynomial $p(x)$ has distinct roots. Let Γ be a root of $p(x)$ such that $|\Gamma| > |\gamma|$ where γ is any other root of $p(x)$ with the exception of $\gamma = \overline{\Gamma}$ when Γ is not real.

If $\Gamma > 0$ and $c > 0$, then $\{u_n\}_{n=0}^{\infty}$ is a.p., and $\{u_n\}_{n=0}^{\infty}$ has infinitely many negative terms otherwise.

Proof: From (2), we have

$$u_n = \sum_{i=1}^{s} c_{r_i} r_i^n + \sum_{i=s+1}^{t} 2\text{Re}(c_{\alpha_i} \alpha_i^n)$$

or, assuming $\Gamma \in \mathbb{R}$, and letting $c = c_\Gamma$,

(3) $$u_n = c\Gamma^n(1 + o(1)).$$

It is clear from (3) that $\Gamma > 0$, $c > 0$, will insure that $\{u_n\}_{n=0}^{\infty}$ is a.p. and that $c < 0$ or $\Gamma < 0$ will produce infinitely many negative terms.

If Γ is not real, we obtain from (3) by use of Euler's formula

(4) $$u_n = |c||\Gamma|^n(\cos(\arg c + n \arg \Gamma))(1 + o(1)).$$

From Lemma 1, we conclude that $\{u_n\}$ has infinitely many negative terms.

The examples at the beginning of the article serve as a simple illustration. The sequence $\{w_n\}$ has as its Binet formula $w_n = 1^n + 1^n + (1.01)^n$, which is clearly positive for all $n \geq 0$. However, the roots associated with $\{u_n\}$ are $1, 1 \pm \sqrt{-1}/10$. Thus the root called Γ in Theorem 1 is $1 + \sqrt{-1}/10$ which is not real. Therefore $\{u_n\}$ has infinitely many positive and infinitely many negative terms.

We now discuss the case of $p(x)$ having s distinct roots of greatest magnitude $|\Gamma|$. Again appealing to the Binet formula, we have

(5) $$u_n = |\Gamma|^n\{c_1 + (-1)^{f(n)}c_2 + c_3 \cos(\lambda_3 + n\theta_3) + \cdots + c_s \cos(\lambda_s + n\theta_s) + o(1)\}, \quad c_i \in \mathbb{R}.$$

By letting $f(n) = n$ or $n + 1$ we may assume $c_2 \geq 0$. Also, as the cosine is an even periodic function, $\cos(\lambda + n\theta) = -\cos((\lambda + \pi) + n\theta)$. Thus when necessary, we may replace $\cos(\lambda_i + n\theta_i)$ by $\cos((\lambda_i + \pi) + n\theta_i)$ and thereby allow us to assume $c_i \geq 0$, $3 \leq i \leq s$.

Theorem 2: Let u_n be as in (5). If $c_1 - c_2 - \cdots - c_s > 0$, then $\{u_n\}_{n=0}^{\infty}$ is a.p.

Proof: Let $\eta > 0$ be such that

$$c_1 - \sum_{i=2}^{s} c_i > \eta > 0.$$

We have

$$u_n = |\Gamma|^n(c_1 + (-1)^{f(n)}c_2 + c_3 \cos(\lambda_3 + n\theta_3) + \cdots + c_3(\cos(\lambda_s + n\theta_s) + g(n))$$

where $g(n)$ is $o(1)$. Choose N so large that for $n > N$, $|g(n)| < \eta/2$. Then for $n > N$,

$$u_n = |\Gamma|^n(c_1 + (-1)^{f(n)}c_2 + c_3 \cos(\lambda_3 + n\theta_3) + \cdots + c_3 \cos(\lambda_3 + n\theta_3) + g(n)) \geq |\Gamma|^n(\eta - \eta/2) = |\Gamma|^n(\eta/2) > 0.$$

Thus $\{u_n\}_{n=0}^{\infty}$ is a.p.

It may be noted that Theorem 2 is the best possible in the following sense: Let

$$u_n = -\frac{1}{2}u_{n-1} + u_{n-2} - \frac{1}{2}u_{n-3}.$$

We have

$$u_n = c_1(1)^n + c_2(-1)^n + c_3\left(-\frac{1}{2}\right)^n \text{ where } c_3\left(-\frac{1}{2}\right)^n \text{ is } o(1).$$

If we choose $c_1 = c_2 = 1$, then $c_1 - c_2 = 0$. As every other term of u_n is negative, $\{u_n\}_{n=0}^{\infty}$ is not a.p. Thus the condition $c_1 - c_2 - \cdots - c_3 > 0$ may not in general be relaxed.

However, upon examining specific cases, it is often possible to improve Theorem 2. For example, if θ_i is a rational multiple of 2π, then

$$\{\cos(\lambda_i + n\theta_i)\}_{n=0}^{\infty}$$

is periodic. Letting

$$\alpha_i = \left|\min_n\{\cos(\lambda_i + n\theta_i)\}_{i=0}^{n}\right|$$

we may use, in Theorem 2, the condition

$$c_1 - c_2 - \cdots - \alpha_i c_i - \cdots - c_s > 0.$$

Thus it is evident how improvements of Theorem 2 can be made when more is known about the roots of the characteristic polynomial.

We now consider the special case of second-order linear recurrences which are completely characterized by the following theorem.

Theorem 3: Let $u_n = au_{n-1} + bu_{n-2}$, $a,b \in \mathbb{R}$, be a second-order linear recurrence. Let

$$\alpha_1 = \frac{a + \delta}{2},\ \alpha_2 = \frac{a - \delta}{2}$$

be the roots of $p(x) = x^2 - ax - b$ where $\delta = \sqrt{a^2 + 4b}$. $\{u_n\}_{n=0}^{\infty}$ is a.p. if and only if $\delta \in \mathbb{R}$ and either

(i) $a = 0$, $u_0 > 0$, $u_1 > 0$

or

(ii) $a > 0$, $2u_1 > (a - \delta)u_0$

where u_0, u_1 are the initial values.

Proof: *Case 1*.—Suppose that δ is not real. Since $\alpha_2 = \overline{\alpha}_1$, Theorem 1 applies with $\Gamma = \alpha_1 \not> 0$. Thus $\{u_n\}_{n=0}^{\infty}$ is not a.p.

Case 2.—$a < 0$. The root of largest absolute value is α_2, and $\alpha_2 < 0$. By Theorem 1, $\{u_n\}_{n=0}^{\infty}$ has infinitely many negative terms.

Case 3.—$a = 0$. The recurrence becomes $u_n = bu_{n-2}$ and the roots of $p(x)$ are $\pm\delta/2$. From the Binet formula, we have

$$u_n = c_1\left(\frac{\delta}{2}\right)^n + c_2\left(-\frac{\delta}{2}\right)^n = \left(\frac{\delta}{2}\right)^n(c_1 + (-1)^n c_2).$$

If suffices to show $c_1 + c_2 > 0$ and $c_1 - c_2 > 0$. $u_0 = c_1 + c_2$ so we must have $u_0 > 0$. $u_1 = \frac{\delta}{2}(c_1 - c_2)$ so that $\frac{2}{\delta}u_1 = c_1 - c_2 > 0$. As $\frac{2}{\delta} > 0$ we have $u_1 > 0$.

Case 4.—$a > 0$. The largest root in absolute value is α_1, and $\alpha_1 > 0$. From Theorem 1 it suffices to show $c_1 > 0$. Using Cramer's Rule, we have

$$c_1 = \frac{u_1 - u_0\alpha_2}{\alpha_1 - \alpha_2}.$$

Since $\alpha_1 - \alpha_2 > 0$, we require that $u_1 - u_0\alpha_2 > 0$ or $2u_1 > u_0(a - \delta)$.

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