$$\overline{\Omega}^2 W_n = (2 - \Omega)^2 W_n = (4 - 4\Omega + \Omega^2) W_n = 4W_n - 4\Omega W_n + \Omega^2 W_n.$$

This result can be verified directly through substitution by (1), (9), and (12), recalling that $P_n = \Omega W_n$ and $\overline{\Omega}^2 W_n = \overline{\Omega^2} W_n$. Once again, by induction on $\lambda$, it is easily shown that

$$\overline{\Omega}^\lambda W_n = (2 - \Omega)^\lambda W_n. \tag{28}$$

It remains open to conjecture whether an examination of various permutations of the operators $\Omega$ and $\overline{\Omega}$, together with the operator $\Delta$ (defined in [4]) and its conjugate $\overline{\Delta}$, will lead to further interesting relationships for higher-order quaternions.

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# ON THE CONVERGENCE OF ITERATED EXPONENTIATION—II*


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In a previous paper [1], we have discussed the properties of the function $f(x)$ defined as:

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}} \tag{1}$$

and a generalization of $f(x)$, namely [2, 3],

$$F_n(x) = g_1(x)^{g_2(x)^{g_3(x)^{\cdot^{\cdot^{\cdot^{g_n(x)}}}}}} = \mathop{\Xi}_{j=1}^{n} g_j(x), \tag{2}$$

where the $g_j(x)$ are functions of a positive real variable $x$, and the symbol $\Xi$ is used to denote the iterated exponentiation [4]. For both (1) and (2), the


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ordering of the exponentiations is important; here and throughout this paper, we mean a bracketing order "from the top down" e.g., for (2), $g_{n-1}$ raised to the power $g_n$, followed by $g_{n-2}$ raised to the resulting power, all the way down to $g_1$. It was shown in [1] that $f$ converges as the number of $x$'s in (1) increases for $x$ from $e^{-e} \cong 0.065988\ldots$ to $e^{1/e} \cong 1.444668\ldots$ . For $x > e^{1/e}$, $f$ is divergent, and for $x < e^{-e}$, the function $f$ is "dual convergent," i.e., it converges to two different values according as the number of $x$'s [or $n$ in (2)] is even or odd. If the number of $x$'s is even, one obtains a curve of $f(x)$ which increases from $1/e$ at $x = e^{-e}$ to $f = 1$ at $x = 0$, and if the number of $x$'s is odd, one obtains a second curve of $f(x)$ which decreases from the unique value $f(e^{-e}) = 1/e = 0.36788$ to $f(0) = 0$ at $x = 0$. Typical values of the limiting $f(x)$ in the region $0 < x < e^{-e}$ are: $f(0.02) \cong 0.03146$ (odd number $n$ of $x$'s) and $f(0.02) \cong 0.88419$ (even $n$); also $f(0.04) \cong 0.08960$ (odd $n$), 0.74945 (even $n$); $f(0.06) \cong 0.21690$ (odd $n$), 0.54323 (even $n$). The property of dual convergence has been shown in [1] and [3] to be a general property of the function $F_n(x)$ of (2), when $g_j(x)$ is a decreasing function of $j$ for fixed $x$, e.g., the function $g_j(x) = x/j^2$, for which $F_n(x)$ is shown in Fig. 3 of [1].

In the present paper we consider a particularly simple generalization of the function $f(x)$, namely the function $F(x, y)$ defined as:

$$(3) \qquad F(x, y) = x^{y^{x^{y^{\cdot^{\cdot^{\cdot^{x^y}}}}}}}$$

where an infinite number of exponentiations is understood, and $x$ is at the bottom of the "ladder." Thus, $F(x, y)$ corresponds to the limit of $F_n(x)$ as $n \to \infty$ in (2), where $g_j(x) \equiv x$ for $j$ = odd, and $g_j(x) \equiv y$ for $j$ = even. Both $x$ and $y$ are assumed to be positive (real) quantities. Depending upon the values of $x$ and $y$, $F(x, y)$ can be monoconvergent, dual convergent, or divergent. For the special case $x = y$, $F(x, x) = f(x)$ of (1), which is monoconvergent in the range $e^{-e} < x < e^{1/e}$, as discussed above. Also, we have $F(x, 1) = x$, $F(1, y) = 1$; $F(x, 0) = 1$, $F(0, y) = 0$, for finite $x$ and $y$. We now consider the case where $x > 1$. We also expand the definition of $F(x, y)$ to include the function

$$(4) \qquad F(y, x) \equiv F'(x, y) = y^{x^{y^{\cdot^{\cdot^{\cdot^{x^y}}}}}}$$

where $y$ is at the bottom of the "ladder."

By enlarging the definition of $F(x, y)$ to include the function $F(y, x)$, we obtain the following three convergence possibilities:

1. *Dual convergence*, when $F(x, y)$ converges to a well-defined value regardless of whether the number of $x$'s in the "ladder" is even or odd. In this case $F(y, x)$ also converges to a well-defined value. Because of the total of two values involved $[F(y, x) \neq F(x, y)]$, we have called this possibility "dual convergence."

2. *Quadriconvergence*, when $F(x, y)$ converges to *two* well-defined values depending upon whether the number of $x$'s in the "ladder" is even or odd. In this case $F(y, x)$ also converges to two well-defined values, again depending upon whether the number of $x$'s and $y$'s in the "ladder" is even or odd. Because of the total of four values of the functions $F(x, y)$ and $F(y, x)$, we have called this possibility "quadriconvergence." However, it should be realized that the quadriconvergence corresponds to the dual convergence of both $F(x, y)$ and $F(y, x)$ in the sense defined in [1] and [3].

3. *Divergence*, in which case both $F(x, y)$ and $F(y, x)$ diverge as the number of $x$'s and $y$'s in (3) and (4) is increased indefinitely. In Figs. 1 and 3 and in Table 1, we have abbreviated dual convergence as D.C., quadriconvergence as Q.C., and divergence as Div.

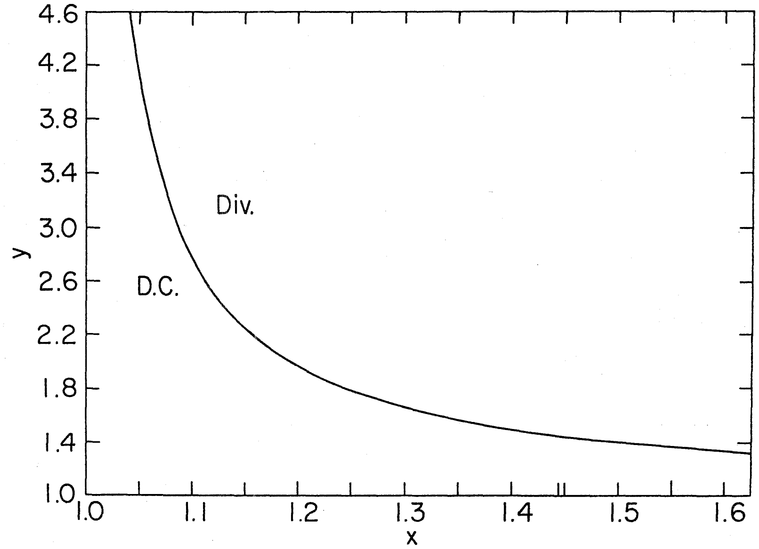


*Fig. 1.* The curve of the limiting $y$ value $y_{\lim}$ as a function of $x$ for $x > 1$, such that for $y > y_{\lim}$, the function $F(x, y)$ is divergent and for $y \leq y_{\lim}$, $F(x, y)$ is dual convergent, i.e., it converges to two values $F_1$ and $F_2$ depending upon whether $x$ or $y$ is at the bottom of the "ladder" in (3) and (4). The point $x = e^{1/e} = 1.444668$, for which $y_{\lim} = x$ has been marked on the abscissa axis.

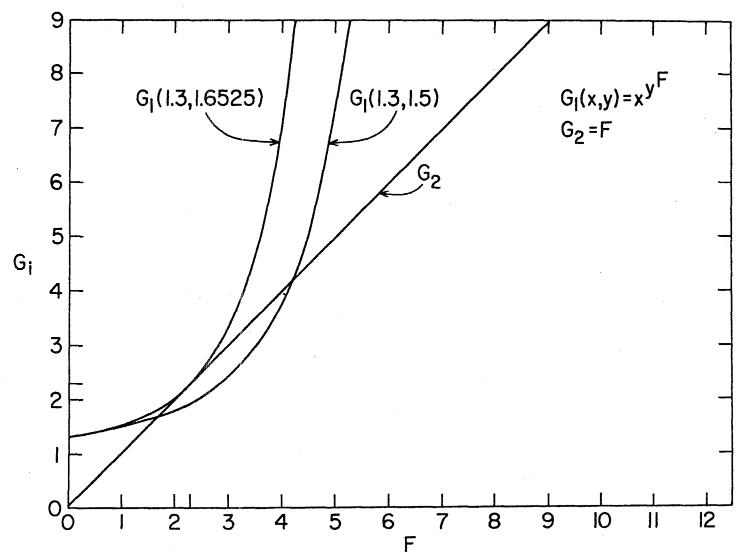


*Fig. 2.* The functions $G_1 = x^{y^F}$ and $G_2 = F$ plotted vs $F$. The two curves of $G_1$ pertain to $x = 1.3$, $y = 1.5$, and $x = 1.3$, $y = 1.6525$, respectively. The curve of $G_1(1.3, 1.5)$ intersects the 45° line $G_2 = F$ at the two points $F^{(1)} = 1.679$ and $F^{(2)} = 4.184$, whose significance is explained in the text. The curve of $G_1(1.3, 1.6525)$ is tangent to the $G_2 = F$ line at $F = 2.304$. Note that 1.6525 is the value of $y_{\lim}$ pertaining to $x = 1.3$.

*Table 1.* A listing of the values of $F(x, y)$ for several illustrative choices of $x$ and $y$. The third column indicates whether the function $F(x,y)$ is dual convergent or quadriconvergent. For dual convergence, the two values of $F_1$ and $F_2$ are listed, which correspond to $F$ of (3) and $F'$ of (4), with $x$ at the bottom of the "ladder" and $y$ at the bottom of the "ladder," respectively. Thus we have $F_1 = x^{F_2}$ and $F_2 = y^{F_1}$. For the cases of quadriconvergence, four values $F_1$, $F_2$, $F_3$, and $F_4$ are listed, where the relations between the $F_i$ are given by (23). The last column of the table lists the value of $y_{\text{lim}}$ for the $x$ value considered. For $0 < x < 1$, $y_{\text{lim}}$ defines the boundary between the regions of dual convergence and quadriconvergence (see Fig. 3). For $x > 1$, $y_{\text{lim}}$ defines the boundary between the dual convergence region and the region where $F(x, y)$ is divergent (see Fig. 1).

| $x$ | $y$ | Conv. | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $y_{\text{lim}}$ |
|---|---|---|---|---|---|---|---|
| 0.2 | 60 | D.C. | 0.09398 | 1.4693 | | | 107.0 |
| 0.2 | 150 | Q.C. | 0.14901 | 2.1099 | 0.03352 | 1.1829 | 107.0 |
| 0.2 | 10,000 | Q.C. | 0.19988 | 6.3028 | $3.93 \times 10^{-5}$ | 1.00036 | 107.0 |
| 0.4 | 20 | D.C. | 0.19414 | 1.7889 | | | 24.02 |
| 0.4 | 30 | Q.C. | 0.31046 | 2.8747 | 0.07179 | 1.2766 | 24.02 |
| 0.4 | 1,000 | Q.C. | 0.40000 | 15.849 | $4.93 \times 10^{-7}$ | 1.0000 | 24.02 |
| 0.7 | 10 | D.C. | 0.40447 | 2.5379 | | | 15.16 |
| 0.7 | 25 | Q.C. | 0.65509 | 8.2371 | 0.05297 | 1.1859 | 15.16 |
| 0.7 | 1,000 | Q.C. | 0.70000 | 125.89 | $3.16 \times 10^{-20}$ | 1.0000 | 15.16 |
| 0.9 | 15 | D.C. | 0.59224 | 4.9719 | | | 21.55 |
| 0.9 | 30 | Q.C. | 0.82743 | 16.681 | 0.17248 | 1.7979 | 21.55 |
| 0.9 | 1,000 | Q.C. | 0.90000 | 501.19 | $1.167 \times 10^{-23}$ | 1.0000 | 21.55 |
| 1.05 | 3.80 | D.C. | 1.3379 | 5.9658 | | | 4.1232 |
| 1.10 | 2.40 | D.C. | 1.3732 | 3.3274 | | | 2.7497 |
| 1.20 | 1.80 | D.C. | 1.5914 | 2.5482 | | | 1.9514 |
| 1.30 | 1.50 | D.C. | 1.6792 | 1.9756 | | | 1.6527 |
| 1.40 | 1.46 | D.C. | 2.1154 | 2.2267 | | | 1.4940 |

In this connection, it should be pointed out that for $x \neq y$, if there is convergence, the minimum number of values obtained is two, namely $F$ and $F'$, and we have the following obvious relations:

$$F(x, y) = x^{F'(x,y)}, \tag{5}$$

$$F'(x, y) = y^{F(x,y)}. \tag{6}$$

The curve of $y_{\text{lim}}$ vs $x$ for $x > 1$ is shown in Fig. 1, where $y_{\text{lim}}$ is the limiting value of $y$ for convergence. This curve was obtained from the following equation derivable directly from (3):

$$F(x, y) = x^{y^{F(x,y)}}. \tag{7}$$

To obtain $y_{\text{lim}}$ as a function of $x$, the following procedure was employed using a Hewlett-Packard calculator. Consider the plane $(F, G)$, with $F$ along the abscissa and $G$ along the ordinate. For a given value of $x$ and a trial value of $y$, the curve $G_1 = x^{y^F}$ was plotted as a function of $F$. This is an increasing function of $F$, since $x > 1$ and $y > 1$. Thus, for $F = 0$, $y^F = 1$, $G_1 = x$, and the curve is concave upward as $F$ is increased to positive values. The intersection of this upward curve with the straight line $G_2 = F$ is then searched for. If $y$ is too large and, hence, if $x^y$ is too large, the curve $G_1$ will not intersect the 45° line $G_2 = F$ (which starts at zero for $F = 0$). Thus, this value of $y$ will be larger than $y_{\lim}$, and the function $F(x, y)$ diverges, and of course also $F'(x, y)$. If $y$ is made appreciably smaller, the curve of $G_1$ will rise more slowly and will generally intersect the 45° line $G_2 = F$ at two values of $F$. It can be shown that the lower value of $F$ gives the correct $F$ as obtained by continued exponentiation, and the corresponding value of $F'$ is given by

$$F' = y^F.$$

Finally, for a certain intermediate value of $y$, the curve $x^y$ vs $F$ will be just tangent to the 45° line $G_2 = F$. This value of $y$ is the limiting value $y_{\lim}$, which we have plotted in Fig. 1 as a function of $x$. An illustration of the possible relationships in the $G$ vs $F$ plane is shown in Fig. 2, for the case $x = 1.3$, for which $y_{\lim} = 1.6525$. Thus, Fig. 2 shows that the derivative of $G_1$ at the tangent point must be +1. Thus:

$$\left.\frac{dx^{y^F}}{dF}\right|_F = +1. \tag{8}$$

This condition, together with the equation

$$x^{y^F} = F, \tag{9}$$

can be used to derive equations for $x$ and $y$, given the assumed value of $F$. We obtain, from (8),

$$\frac{d}{dF}x^{y^F} = \frac{d}{dF}\exp\{\log x[\exp(F \log y)]\} = F\frac{d}{dF}\{\log x[\exp(F \log y)]\} = +1, \tag{10}$$

whence:

$$\frac{1}{F} = \log x \frac{d}{dF}[\exp(F \log y)] = \log x \log y \exp(F \log y). \tag{11}$$

But from (9), we find

$$F = x^{y^F} = x^{\exp(F \log y)} = \exp[\log x \exp(F \log y)], \tag{12}$$

so that

$$\log F = \log x \exp(F \log y). \tag{13}$$

Upon dividing (11) by (13), we obtain

$$\frac{1}{F \log F} = \log y, \tag{14}$$

which gives

$$y = \exp(1/F \log F). \tag{15}$$

In order to obtain the corresponding equation for $x$, we note that from (12) and (15),

$$\log F = \log x\, y^F = \log x \exp(1/\log F), \tag{16}$$

which gives:

$$\log x = \log F \exp(-1/\log F), \tag{17}$$

$$x = \exp[\log F \exp(-1/\log F)] = \exp[\log F/\exp(1/\log F)]. \tag{18}$$

For the case where one of the quantities, say $x$, is less than 1, but where $y$ can be large, and still keeping $y > 1$, we have a somewhat different situation. In this case, the function $G_1 = x^{y^F}$ is a decreasing function of $F$, starting at $G_1 = x$ for $F = 0$ and going down to $x^y$ ($< x$) at $F = 1$. Thus, the curve of $G_1$ vs $F$ will always intersect the 45° line $G_2 = F$ at a value of $F < 1$. It can then be shown that the functions $F$ and $F'$ must be quadriconvergent if the negative slope $dx^{y^F}/dF$ at $x^{y^F} = F$ is algebraically smaller than -1. Thus, the limiting curve of $y_{\text{lim}}$ vs $x$ which separates the regions of dual and quadriconvergence is obtained from the following pair of equations:

$$\left.\frac{dx^{y^F}}{dF}\right|_F = -1, \tag{19}$$

$$x^{y^F} = F. \tag{20}$$

Thus, if the slope $\left(dx^{y^F}/dF\right) < -1$, we will have quadriconvergence, whereas for $\left(dx^{y^F}/dF\right) > -1$, we will have dual convergence.

Now we note that (19) and (20) are remarkably similar to (8) and (9), the only difference being the change of sign in (19) as compared to (8). We thus obtain the following equations for $x$ and $y$ for the limiting curve (i.e., $y = y_{\text{lim}}$):

$$x = \exp[\log F \ \exp(1/\log F)], \tag{21}$$

$$y = \exp(-1/F \ \log F). \tag{22}$$

By means of these equations, we have obtained the plot of $y$ vs $x$ of Fig. 3.

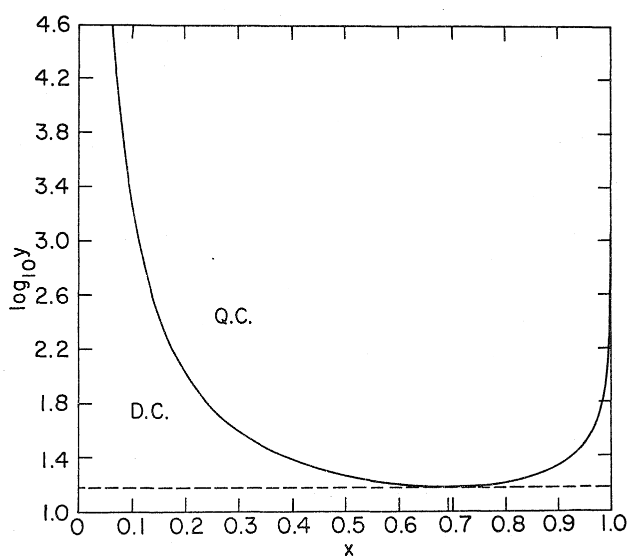


*Fig. 3. The curve of $\log_{10} y_{\text{lim}}$ as a function of $x$ for $0 < x < 1$. For $y \le y_{\text{lim}}$, the function $F(x, y)$ is dual convergent, i.e., it converges to two values $F_1$ and $F_2$, depending on whether $x$ or $y$ is at the bottom of the "ladder" in (3) and (4). For $y > y_{\text{lim}}$, $F(x, y)$ is quadriconvergent, i.e., it converges to two values each for both $x$ and $y$ at the bottom of the "ladder" in (3) and (4); thus, it converges to four values altogether [see (23) and (24)]. The dashed horizontal line $\log_{10} y = \log_{10}(e^e) = \log_{10} 15.15421$ is tangent to the curve at the point $x = e^{-1/e} = 0.692201$.*

By letting $F' = y^F$ vary from $F' = 1$ to large $F'$, we cover the range $x = 0$ to $x = 1$. (Note that $y > 1$ is assumed.) The regions of dual convergence and quadriconvergence are indicated as D.C. and Q.C., respectively. We note that regardless of $x$ in the range 0 to 1 the functions $F$ and $F'$ will each converge to a single value, provided that $y < e^e \cong 15.154$. The line $y = e^e$ is marked as a dashed line and the curve of $y$ vs $x$ is tangent to this line at the point $x = e^{-1/e} \cong 0.6922$. This value of $x$ is just the reciprocal of the value $x' = e^{1/e}$ which is the limit of convergence of the function $f(x) = F(x, x)$ which has been discussed in [1] - [3]. We also note that the minimum value of $y_{\text{lim}}$ for $x < 1$, namely $y_{\text{lim}} = e^e$, is just the reciprocal of the value $x = e^{-e} = 0.065988$, below which the function $f(x)$ becomes dual convergent, as has been shown in [1]. The value of $f(x = e^{-e})$ is $1/e$. The curve of $y_{\text{lim}}$ vs $x$ is asymptotic to the vertical lines $x = 0$ and $x = 1$ in Fig. 3.

Values of the functions $F(x, y)$ and $F'(x, y)$ have been calculated by means of iterated exponentiation on a Hewlett-Packard calculator. We have considered a large number of combinations $(x, y)$, both on the limiting curve $(x, y_{\text{lim}})$ where the convergence is slow and away from the limiting curve $(x, y_{\text{lim}})$ where the convergence is much faster. (The computing program was designed to carry out up to 1600 exponentiations, if necessary.) A few typical values exhibiting both dual and quadriconvergence have been tabulated in Table 1. For the reader's convenience, we have listed the value of $y_{\text{lim}}$ pertaining to the $x$ value in each entry. Also, the notation D.C. or Q.C. has been included.

For the case of quadriconvergence, we have listed in Table 1 four values denoted by $F_1$, $F_2$, $F_3$, and $F_4$. In order to make the identification of the $F_i$ $(i = 1 - 4)$ with the functions $F(x, y)$ and $F'(x, y)$ introduced above in (3) and (4), we note that we have the following relations:

$$y^{F_1} = F_2,\ x^{F_2} = F_3,\ y^{F_3} = F_4,\ x^{F_4} = F_1, \tag{23}$$

so that we can write

$$F_1 = F_a,\ F_2 = F_a',\ F_3 = F_b,\ F_4 = F_b'. \tag{24}$$

Both $F_1$ and $F_3$ are functions of the type $F$ with $x$ at the bottom of the "ladder" [see (3)], and they are therefore denoted by $F_a$ and $F_b$, respectively. Similarly, $F_2$ and $F_4$ are functions of the type $F'$ with $y$ at the bottom of the "ladder" [see (4)], and they are therefore denoted by $F_a'$ and $F_b'$, respectively. In view of (23) and (24), we see that the quadriconvergence for $y > y_{\text{lim}}$ (and $x < 1$) is actually the analog of the dual convergence observed in [1] and [3] for functions of one variable $(x)$ only, since the functions $F_a$ and $F_b$ which have the same definition take on two different values, and similarly for $F_a'$ and $F_b'$.

For the case of dual convergence of $F(x, y)$ and $F'(x, y)$ which occurs when $y \leq y_{\text{lim}}$, the two functions $F_1$ and $F_2$ of Table 1 can be simply identified as $F_1 = F$ and $F_2 = F'$ of (3) and (4).

In Table 1, we have included a few cases with $y$ very large (for $x < 1$), namely, $y = 10{,}000$ for $x = 0.2$ and $y = 1{,}000$ for $x = 0.4$, 0.7, and 0.9. The reason is that, in the limiting case of large $y$, the following equations hold to a very high accuracy, as is shown by the entries in Table 1:

$$F_1 \approx x,\ F_2 \approx y^x,\ F_3 \approx 0,\ F_4 \approx 1. \tag{25}$$

The above equations can be derived very simply by noting that starting with a value $F_1 = x$, we have $F_2 = y^x$, and if $y^x$ is large enough, $F_3 = x^{(y^x)}$ will be very small (i.e., $\approx 0$) for $x < 1$, and hence, $F_4 \cong y^0 = 1$, and the next value to be denoted by $F_5$ is: $F_5 \approx x^1 = x$, i.e., $F$ has the value assumed above for $F_1$, so that the four equations of (25) are mutually consistent, provided that $y^x \gg 1$, so that $x^{(y^x)} \approx 0$.

Before leaving this discussion of the functions $f$ and $F$, we wish to point out an interesting property. First, considering the function $F$ at the tangency point $x = e^{-1/e}$ (see Fig. 3), for the two values of $F$ at $x = e^{-1/e} = 0.692200$, $y = e^e = 15.1542$, we find $F'(x, y) = e$, and $F(x, y) = x^{F'} = 1/e$. Furthermore, for the function $f(x)$ at the point $x = e^{-e} = 0.065988$, we find the single value $f(x = e^{-e}) = 1/e$, whereas at the other extreme of the region of convergence, namely, $x = e^{1/e}$, we find $f(x) = e$. Thus, the six quantities

$$e,\ 1/e,\ e^{1/e},\ e^{-1/e},\ e^e, \text{ and } e^{-e}$$

are directly involved in the results obtained for the functions $f(x)$ and $F(x, y)$ at certain special points $x$ and $y$.

Finally we will consider a generalization of the functions $f(x)$ and $F(x, y)$ to be denoted $f_N(x)$ and $F_N(x, y)$, respectively. We first define $f_N(x)$ by the equation

$$f_N(x) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x^N}}}}}}, \tag{26}$$

where $N$ is an arbitrary positive quantity, and we are interested in the limit of an infinite number of $x$'s in the "ladder." Again, the bracketing order is as usual "from the top down." Now for $N = x$, we find $f_x(x) = f(x)$ as before. It can be shown that for $x > 1$, if $N$ is too large, the function $f_N(x)$ diverges even though $x$ lies in the range $1 < x < e^{1/e}$ for which the simpler function $f(x)$ converges. In order to obtain the limitation on $N$, we consider the plane of $G$ vs $f$ as shown in Fig. 4. The line $G_2 = f$ is the 45° straight line in this figure. In addition, we have plotted the function $G_1(x) = x^f$ for two different values of $x$, namely, $x = 1.35$ and $x = e^{1/e} = 1.444668$. For $x = e^{1/e}$, $G_1(x)$ is just tangent to the straight line $G_2 = f$ at $f = e$. However, for $x = 1.35$, $G_1(x)$ intersects the line $G_2 = f$ at two values of $f$, namely, $f^{(1)} = 1.6318$ and $f^{(2)} = 5.934$. The value $f^{(1)}$ corresponds to the simple function $f(x = 1.35)$. We now note that in the region of $f$, $1.6318 < f < 5.934$, we have $1.35^f < f$, as shown by Fig. 4. It is therefore easy to show that if $N \leq 5.934$, the function $f_N(x)$ of (26) converges simply to the value $f(x = 1.35) = 1.6318$. On the other hand, for $N > 5.934$, we have $1.35^N > N$, so that as we go down the "ladder" of (26), progressively larger results are obtained and the function $f_N(1.35)$ diverges in this case even though $f(1.35)$ converges, since $x < e^{1/e}$. The value $f^{(2)}$, which is the limiting value for $N$, corresponds to the dashed part of the curve of $x$ vs $f(x)$ in Fig. 1 of [1], which we had labeled at that time as "not meaningful" for the function $f(x)$. As can be seen from this figure, $f^{(2)}(x)$ increases rapidly with decreasing $x$ until it becomes infinite as $x \to 1$. Typical values of $f^{(2)}(x)$, as obtained from the equations

$$x^f = f \tag{27}$$

$$\log x = \log f/f, \tag{28}$$

are as follows:

$$f^{(2)}(1.4) = 4.41,\ f^{(2)}(1.3) = 7.86,\ f^{(2)}(1.2) = 14.77,\ f^{(2)}(1.15) = 22.17,$$
$$f^{(2)}(1.1) = 38.2,\ f^{(2)}(1.05) = 92.95.$$

Thus, for $x = 1.1$, we have

$$f_N(1.1) = f^{(1)}(1.1) = 1.112, \text{ for } N \leq 38.2, \tag{29}$$

while $f_N(1.1)$ diverges for $N > 38.2$.

It can be easily shown that for $x < 1$, we have $f_N(x) = f(x)$, regardless of the (positive) value of $N$, and, correspondingly, the curve of $x$ vs $f(x)$ in Fig. 2 of [1] does not have a second branch similar to that of Fig. 1.

We now define the function $F_N(x, y)$ as follows:

(30) $$F_N(x, y) = x^{y^{x^{y^{\cdot^{\cdot^{\cdot^{x^{y^N}}}}}}}}.$$

We will examine this function first for the case that both $x$ and $y$ are larger than 1. We assume that $x \leq y$. The situation is then very similar to that for $f_N(x)$. As an illustration, we consider the case where $x = 1.3$, and consider the plane $G_i$ vs $F$, where $G_2 = F$ (45° straight line) and $G_1 = x^{y^F} = 1.3^{y^F}$. For $y = y_{\text{lim}} = 1.6525$, we are at the border between the regions of dual convergence and divergence in Fig. 1. Correspondingly, the curve of $G_1 = 1.3^{1.6525^F}$ is just tangent to the line $G_2 = F$ at the point $F = 2.304$ (see Fig. 2). Now consider the curve $G_1 = 1.3^{1.5^F}$, which has two points of intersection $F^{(1)}$ and $F^{(2)}$ with the line $G_2 = F$. We have:

$$F^{(1)} = 1.679,\ F^{(2)} = 4.184.$$

For $F^{(1)} < F < F^{(2)}$, we find that $G_1(1.3, 1.5) = 1.3^{1.5^F} < F$. Therefore, it can be concluded in the same manner as for $f_N(x)$ that $F_N(1.3, 1.5)$ converges to the value $F(1.3, 1.5)$ for $N \leq 4.184$, while for $N > 4.184$, $F(1.3, 1.5)$ diverges. Thus, for $(x, y)$ with $y < y_{\text{lim}}$, the roots of the equation

(31) $$x^{y^F} - F = 0,$$

determine both the value of $F$ $(=F^{(1)})$ and of $N_{\max}$, such that for $N \leq N_{\max}$, the modified function $F_N(x, y)$ converges to the value of $F(x, y)$. Here $N_{\max} = F^{(2)}$. Of course, for $y = y_{\text{lim}}$, we have $F^{(1)} = F^{(2)}$ (point of tangency), and $N_{\max} = F^{(1)} = F^{(2)}$. As an example, for $x = 1.3$, $y = 1.6525$, the tangency occurs at $F = 2.304$ in Fig. 4, and we have convergence of $F(1.3, 1.6525)$ to the value $F = 2.304$, provided that $N \leq 2.304$.

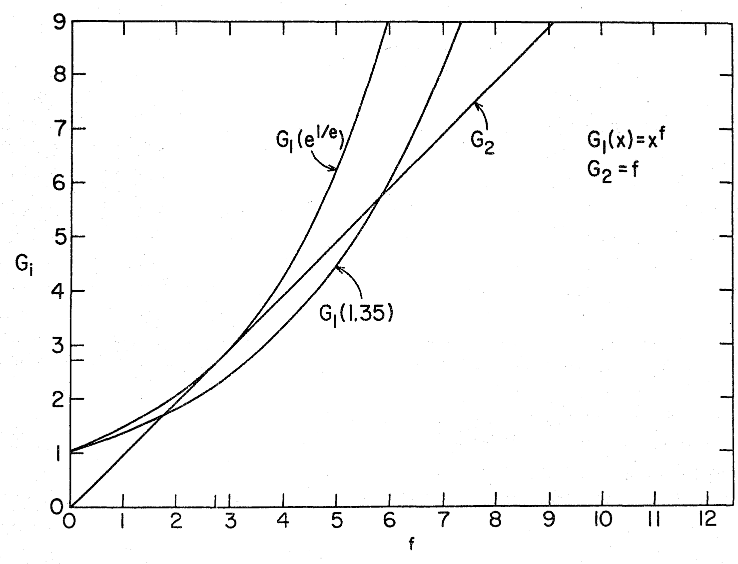


*Fig. 4. The functions $G_1 = x^f$ and $G_2 = f$ plotted vs $f$. The two curves of $G_1$ pertain to the $x$ values $x = 1.35$ and $x = e^{1/e} = 1.444668$. The curve of $G_1(1.35)$ intersects the 45° line $G_2 = f$ at the two points $f^{(1)} = 1.6318$ and $f^{(2)} = 5.934$, whose significance is explained in the text. The curve of $G_1(e^{1/e})$ is tangent to the $G_2 = f$ line at $f = e$ (see [1]).*

When either $x$ or $y < 1$ (or both $x$ and $y < 1$), it is easily shown that the function $F_N(x, y) = F(x, y)$, regardless of the value of $N$. Thus, assume that

$x < 1$, but $y > 1$. Then, if $N$ is arbitrarily large, $y^N$ will be still larger, i.e., $y^N = N'$ where $N' > N$. The next step in the calculation of $F(x, y)$ involves raising $x$ to the power $N'$. For $N'$ very large, we find $x^{N'} \sim 0$, followed by $y^0 = 1$, and $x^1 = x$. This proves that $F_N(x, y) = F(x, y)$ regardless of the value of $N$. Note that for $N$ very small, we have $y^N \sim 1$, followed by $x^{y^N} \cong x^1 = x$, independently of $N$.

The preceding argument involving $F_N$ can also be used to prove the following theorem, when a similar function $H$ of more than two variables is involved. Here we assume that $H$ is a function of the type of $F$ of Eqs. (2) and (3). As an example, we define $H(x, y, z)$ as follows:

$$H(x, y, z) = x^{y^{z^{\cdot^{\cdot^{\cdot^{x^{y^{z}}}}}}}}, \tag{32}$$

where $x$, $y$, $z$ are arbitrary positive quantities. It can be easily shown that if one of the three numbers $x$, $y$, or $z$ is $\leqq 1$, then $H(x, y, z)$ will not diverge (although it may converge to two values for any given value of $x$, $y$, or $z$ at the bottom of the ladder, by virtue of the property of dual convergence introduced in [1] and [3]). To prove the theorem, we assume that $x \leqq 1$, but $y$ and $z > 1$. At the top of the ladder, we obtain $x^{(y^z)}$, where $y^z$ may be arbitrarily large. We will write $y^z = M$. Now $x^M \sim 0$ for $x < 1$ and large $M$. The next step calls for the calculation of $z^{x^M} \sim z^0 = 1$, followed by $y^{z^0} = y$, and so on. It is easily seen that the sequence $H(x, y, z)$ will never diverge provided that $x$, $y$, or $z$ is $\leqq 1$. For the case where $x$, $y$, $z$ are all larger than 1, but do not exceed $e^{1/e}$, we may use the result of [1] to prove that

$$H(x, y, z) \leqq f(e^{1/e}) = e,$$

and thus $H(x, y, z)$ is convergent. On the other hand, if at least one of the triplet $x$, $y$, $z$ is larger than $e^{1/e}$, say $x > e^{1/e}$, whereas the other two lie in the range $1 < (y, z) < e^{1/e}$, then $H(x, y, z)$ will converge or diverge depending on the values of $x$, $y$, $z$ relative to $e^{1/e}$, in the same manner as for $F(x, y)$ (see Fig. 1).

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