ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-454 *Proposed by Charles W. Trigg, San Diego, CA*

In the square array of the nine nonzero digits

$$\begin{matrix} 6 & 7 & 5 \\ 2 & 1 & 3 \\ 9 & 4 & 8 \end{matrix}$$

the sum of the four digits in each 2-by-2 corner array is 16. Rearrange the nine digits so that the sum of the digits in each such corner array is *seven* times the central digit.

B-455 *Proposed by Herta T. Freitag, Roanoke, VA*

Let

$$S_m = \sum_{i=0}^{m} F_{i+1}L_{m-i} \quad \text{and} \quad T_m = 10S_m/(m + 2).$$

Prove that T_m is a sum of two Lucas numbers for $m = 0, 1, 2, \ldots$.

B-456 *Proposed by Albert A. Mullin, Huntsville, AL*

It is well known that any two consecutive Fibonacci numbers are coprime (i.e., their gcd is 1). Prove or disprove: Two distinct Fibonacci numbers are coprime if each of them is the product of two distinct primes.

B-457 *Proposed by Herta T. Freitag, Roanoke, VA*

Prove or disprove that there exists a positive integer b such that the Pythagorean type relationship $(5F_n^2)^2 + b^2 \equiv (L_n^2)^2 \pmod{5m^2}$ holds for all m and n with $m | F_n$.

B-458 *Proposed by H. Klauser, Zürich, Switzerland*

Let T_n be the triangular number $n(n + 1)/2$. For which positive integers k do there exist positive integers n such that $T_{n+k} - T_n$ is a prime?

B-459 *Proposed by E. E. McDonnell, Palo Alto, CA, and J. O. Shallit, Berkeley, CA*

Let g be a primitive root of the odd prime p. For $1 \leq i \leq p - 1$, let a_i be the integer in $S = \{0, 1, \ldots, p - 2\}$ with $g^{a_i} \equiv i \pmod{p}$. Show that

$$a_2 - a_1,\ a_3 - a_2,\ \ldots,\ a_{p-1} - a_{p-2}$$

(differences taken mod $p - 1$ to be in S), is a permutation of $1, 2, \ldots, p - 2$.

SOLUTIONS

Double a Triangular Number

B-430 *Proposed by M. Wachtel, H. Klauser, and E. Schmutz, Zürich, Switzerland*

For every positive integer a, prove that

$$(a^2 + a - 1)(a^2 + 3a + 1) + 1$$

is a product $m(m + 1)$ of two consecutive integers.

Solution by Frank Higgins, Naperville, IL

Noting that

$$(a^2 + a - 1)(a^2 + 3a + 1) + 1 = (a^2 + 2a - 1)(a^2 + 2a),$$

the assertion follows with m the integer $a^2 + 2a - 1$.

Also solved by J. Annulis, Paul S. Bruckman, D. K. Chang, M. J. DeLeon, Charles G. Fain, Herta T. Freitag, Robert Girse, Graham Lord, John W. Milsom, F. D. Parker, Bob Prielipp, A. G. Shannon, Charles B. Shields, Sahib Singh, Lawrence Somer and the proposers.

Making it an Identity

B-431 *Proposed by Verner E. Hoggatt, Jr., San Jose, CA*

For which fixed ordered pairs (h, k) of integers does

$$F_n(L_{n+h}^2 - F_{n+h}^2) = F_{n+4}(L_{n+k}^2 - F_{n+k}^2)$$

for all integers n?

Solution by Paul S. Bruckman, Concord, CA

For any integer m,

$$\begin{aligned} L_m^2 - F_m^2 = (L_m - F_m)(L_m + F_m) &= (F_{m+1} + F_{m-1} - F_m)(F_{m+1} + F_{m-1} + F_m) \\ &= 2F_{m-1} \cdot 2F_{m+1} = 4F_{m-1}F_{m+1}. \end{aligned}$$

Hence, the desired identity is equivalent to:

$$F_n F_{n+h+1} F_{n+h-1} = F_{n+4} F_{n+k+1} F_{n+k-1}, \tag{1}$$

which is to hold for some pair (h, k) of integers and for all n. In particular, (1) must hold for $n = 0$ and $n = -4$, which yields:

$$F_4 F_{k+1} F_{k-1} = F_{-4} F_{h-3} F_{h-5} = 0.$$

Since the only term of the Fibonacci sequence that vanishes is F_0, we must have $h \in \{3, 5\}$ and $k \in \{-1, 1\}$, i.e.,

$$(h, k) \in \{(3, -1), (5, -1), (5, 1), (3, 1)\}.$$

Checking out these possibilities, one finds that the unique solution is

$$(h, k) = (3, 1).$$

Also solved by M. D. Agrawal, M. J. DeLeon, Herta T. Freitag, Frank Higgins, John W. Milsom, A. G. Shannon, Charles B. Shields, Sahib Singh, M. Wachtel and the proposer.

Alternating Signs

B-432 *Proposed by Verner E. Hoggatt, Jr., San Jose, CA*

Let

$$G_n = F_n F_{n+3}^2 - F_{n+2}^3.$$

Prove that the terms of the sequence $G_0, G_1, G_2, \ldots$ alternate in sign.

Solution by F. D. Parker, St. Lawrence University, Canton, NY

$$\begin{aligned} G_n &= F_n F_{n+3}^2 - F_{n+2}^3 = F_n(2F_{n+1} + F_n)^2 - (F_n + F_{n+1})^3 \\ &= F_n^3 + 4F_n^2 F_{n+1} + 4F_{n+1}^2 F_n - F_n^3 - 3F_n^2 F_{n+1} - 3F_n F_{n+1}^2 - F_{n+1}^3 \\ &= F_{n+1}^2 F_n + F_{n+1} F_n^2 - F_{n+1}^3 \\ &= F_{n+1}(F_n F_{n+2} - F_{n+1}^2) = (-1)^n F_{n+1}. \end{aligned}$$

Also solved by M. D. Agrawal, Stephan Andres, Paul S. Bruckman, L. Carlitz, M. J. DeLeon, Herta T. Freitag, Frank Higgins, Graham Lord, Bob Prielipp, A. G. Shannon, Sahib Singh, Lawrence Somer and the proposer.

Alternate Definition of a Sequence

B-433 *Proposed by J. F. Peters and R. Pletcher, St. John's University, Collegeville, MN*

For each positive integer n, let q_n and r_n be the integers with

$$n = 3q_n + r_n \quad \text{and} \quad 0 \le r_n < 3.$$

Let $\{T(n)\}$ be defined by

$$T(0) = 1,\ T(1) = 3,\ T(2) = 4, \text{ and } T(n) = 4q_n + T(r_n), \text{ for } n \ge 3.$$

Show that there exist integers a, b, c such that

$$T(n) = \left[\frac{an + b}{c}\right],$$

where $[x]$ denotes the greatest integer in x.

Solution by Sahib Singh, Clarion State College, Clarion, PA

The given arithmetic function $T(n)$ can be defined as

$$T(3t) = 4t + 1;\ T(3t + 1) = 4t + 3;\ T(3t + 2) = 4t + 4$$

or, equivalently,

$$T(n) = [(4n + 5)/3].$$

Hence, $a = 4$, $b = 5$, and $c = 3$.

Also solved by Paul S. Bruckman, M. J. DeLeon, Herta T. Freitag, Frank Higgins, H. Klauser, Graham Lord, A. G. Shannon and the proposers.

Never a Square

B-434 *Proposed by Herta T. Freitag, Roanoke, VA*

For which positive integers n, if any, is $L_{3n} - (-1)^n L_n$ a perfect square?

Solution by A. G. Shannon, New South Wales Institute of Technology, Sydney, Australia

$$L_{3n} - (-1)^n L_n = a^{3n} + b^{3n} - (ab)^n(a^n + b^n) = 5L_n F_n^2,$$

which would be a perfect square if and only if $5|L_n$; but this is impossible for all n.

Also solved by Paul S. Bruckman, Frank Higgins, J.W. Milsom, F. D. Parker, Bob Prielipp, Sahib Singh, Lawrence Somer, M. Wachtel and the proposer.

Restricted Divisors of a Quadratic

B-435 *Proposed by M. Wachtel, H. Klauser, and E. Schmutz, Zürich, Switzerland*

For every positive integer a, prove that no integral divisor of $a^2 + a - 1$ is congruent to 3 or 7 modulo 10.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We begin by observing that since $a^2 + a - 1$ is odd, each of its divisors must be odd. Suppose there is a divisor d of $a^2 + a - 1$ which is congruent to 3 or 7 modulo 10. Then d must have at least one prime divisor p which is congruent to 3 or 7 modulo 10. (If this were not the case, the only primes that could be divisors of d would be 5, primes congruent to 1 modulo 10, and primes congruent to 9 modulo 10. But then d would have to be congruent to 1, 5, or 9 modulo 10.) It follows that $a^2 + a - 1 \equiv 0 \pmod{p}$. Hence, $4a^2 + 4a \equiv 4 \pmod{p}$ so $(2a + 1)^2 \equiv 5 \pmod{p}$. Thus, 5 is a quadratic residue modulo p.

Let q be an odd prime such that $(q, 5) = 1$. Then, by the Law of Quadratic Reciprocity,

$$\left(\frac{5}{q}\right) = \left(\frac{q}{5}\right)$$

and

$$\left(\frac{q}{5}\right) = \begin{cases} \left(\frac{2}{5}\right) = -1 \text{ if } q \equiv 2 \pmod{5}, \\ \left(\frac{3}{5}\right) = -1 \text{ if } q \equiv 3 \pmod{5}. \end{cases}$$

Hence, 5 is a quadratic nonresidue of all odd primes which are congruent to 2 or 3 modulo 5, so 5 is a quadratic nonresidue of every prime congruent to 3 or 7 modulo 10.

This contradiction tells us that no divisor of $a^2 + a - 1$ is congruent to 3 or 7 modulo 10.

Also solved by Paul S. Bruckman, M. J. DeLeon, A. G. Shannon, Sahib Singh, and Lawrence Somer.

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