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THE ANDREWS FORMULA FOR FIBONACCI NUMBERS

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1. George E. Andrews [1] gave the following formulas for the Fibonacci numbers F_n ($F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$) in terms of binomial coefficients $\left((n;\ r) = \binom{n}{r}\right)$:

$$F_n = \sum_j (-1)^j \quad (n - 1;\ [(n - 1 - 5j)/2]), \tag{1.1}$$

$$F_n = \sum_j (-1)^j \quad (n;\ [(n - 1 - 5j)/2]). \tag{1.2}$$

Hansraj Gupta [2] has pointed out that (1.1) and (1.2) can be written, respectively, as

$$F_{2m+1} = S(2m,\ m) - S(2m,\ m - 2), \tag{1.3a}$$

$$F_{2m+2} = S(2m + 1,\ m) - S(2m + 1,\ m - 2) \tag{1.3b}$$

and

$$F_{2m+1} = S(2m + 1,\ m) - S(2m + 1,\ m - 1) \tag{1.4a}$$

$$F_{2m+2} = S(2m + 2,\ m) - S(2m + 1,\ m - 1), \tag{1.4b}$$

where $S(n,\ k) = \Sigma(n;\ j)$, the sum being taken over those j congruent to k modulo 5, and has given inductive proofs of (1.3) and (1.4).

The object of this note is to obtain (1.3) and (1.4) by first finding $S(n, k)$ explicitly in terms of such familiar numbers as

$$\alpha = \frac{1}{2}(1 + \sqrt{5}),\ \beta = \frac{1}{2}(1 - \sqrt{5}).$$

2. We begin by noting that

$$(1 + x)^n = \Sigma(n;\ j)x^j. \tag{2.1}$$

If we put $x = 1, \omega, \omega^2, \omega^3, \omega^4$ into (2.1) in turn (where $\omega = e^{\frac{2\pi i}{5}}$), add the resulting series, and divide by 5, we obtain

$$S(n,\ 0) = \frac{1}{5}(2^n + (1 + \omega)^n + (1 + \omega^2)^n + (1 + \omega^3)^n + (1 + \omega^4)^n). \tag{2.2a}$$

In similar fashion,

$$S(n,\ 1) = \frac{1}{5}(2^n + \omega^4(1 + \omega)^n + \omega^3(1 + \omega^2)^n + \omega^2(1 + \omega^3)^n + \omega(1 + \omega^4)^n), \tag{2.2b}$$

$$S(n,\ 2) = \frac{1}{5}(2^n + \omega^3(1 + \omega)^n + \omega(1 + \omega^2)^n + \omega^4(1 + \omega^3)^n + \omega^2(1 + \omega^4)^n), \tag{2.2c}$$

$$S(n,\ 3) = \frac{1}{5}(2^n + \omega^2(1 + \omega)^n + \omega^4(1 + \omega^2)^n + \omega(1 + \omega^3)^n + \omega^3(1 + \omega^4)^n), \tag{2.2d}$$

$$S(n,\ 4) = \frac{1}{5}(2^n + \omega(1 + \omega)^n + \omega^2(1 + \omega^2)^n + \omega^3(1 + \omega^3)^n + \omega^4(1 + \omega^4)^n). \tag{2.2e}$$

Now, $1 + \omega = 1 + e^{\frac{2\pi i}{5}} = 2\cos\frac{\pi}{5}\cdot e^{\frac{\pi i}{5}} = \alpha e^{\frac{\pi i}{5}}$, and similarly,

$$1 + \omega^2 = -\beta e^{\frac{2\pi i}{5}},$$
$$1 + \omega^3 = -\beta e^{-\frac{2\pi i}{5}},$$
$$1 + \omega^4 = \alpha e^{-\frac{\pi i}{5}},$$

so (2.2a) becomes

$$\begin{aligned} S(n,\ 0) &= \frac{1}{5}(2^n + \alpha^n e^{n\pi i/5} + (-\beta)^n e^{2n\pi i/5} + (-\beta)^n e^{-2n\pi i/5} + \alpha^n e^{-n\pi i/5}) \\ &= \frac{1}{5}(2^n + 2\alpha^n \cos n\pi/5 + 2(-\beta)^n \cos 2n\pi/5). \end{aligned} \tag{2.3a}$$

In similar fashion,

$$S(n,\ 1) = \frac{1}{5}(2^n + 2\alpha^n \cos(n - 2)\pi/5 + 2(-\beta)^n \cos(2n - 4)\pi/5), \tag{2.3b}$$

$$S(n,\ 2) = \frac{1}{5}(2^n + 2\alpha^n \cos(n - 4)\pi/5 + 2(-\beta)^n \cos(2n + 2)\pi/5), \tag{2.3c}$$

$$S(n,\ 3) = \frac{1}{5}(2^n + 2\alpha^n \cos(n + 4)\pi/5 + 2(-\beta)^n \cos(2n - 2)\pi/5), \tag{2.3d}$$

$$S(n,\ 4) = \frac{1}{5}(2^n + 2\alpha^n \cos(n + 2)\pi/5 + 2(-\beta)^n \cos(2n + 4)\pi/5). \tag{2.3e}$$

It follows that, for every k,

$$S(n,\ k) = \frac{1}{5}(2^n + 2\alpha^n \cos(n - 2k)\pi/5 + 2(-\beta)^n \cos(2n - 4k)\pi/5). \tag{2.4}$$

3. Now we are in a position to prove (1.3) and (1.4). We have

$$S(2m,\ m) = \frac{1}{5}(2^{2m} + 2\alpha^{2m} + 2\beta^{2m}),$$

$$S(2m,\ m-2) = \frac{1}{5}\left(2^{2m} + 2\alpha^{2m}\cos\frac{4\pi}{5} + 2\beta^{2m}\cos\frac{2\pi}{5}\right),$$

so

$$\begin{aligned} S(2m,\ m) - S(2m,\ m-2) &= \frac{2}{5}\alpha^{2m}\left(1 - \cos\frac{4\pi}{5}\right) + \frac{2}{5}\beta^{2m}\left(1 - \cos\frac{2\pi}{5}\right) \\ &= \frac{2}{5}\alpha^{2m} \cdot \frac{\sqrt{5}}{2}\alpha + \frac{2}{5}\beta^{2m} \cdot \frac{\sqrt{5}}{2}(-\beta) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2m+1} - \beta^{2m+1}) \\ &= F_{2m+1}, \end{aligned}$$

which is (1.3a). The derivations of (1.3b) and (1.4) from (2.4) are similar, and are omitted.

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SOME CONSTRAINTS ON FERMAT'S LAST THEOREM

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1. INTRODUCTION

The proof of "Fermat's Last Theorem," namely that there are no nontrivial integer solutions of $x^n + y^n = z^n$, where n is an integer greater than 2, is well known for the cases $n = 3$ and 4. We propose to look at some constraints on the values of x, y, and z, if they exist, when $n = p$, an odd prime. The history of the extension of the bounds on is interesting and illuminating [3], as is the development of the theory of ideals from Kummer's attempt to verify Fermat's result for all primes [2].

2. CONSTRAINT ON z

It can be readily established that there is no loss of generality in assuming that $0 < x < y < z$. Since $x \neq y$, $z - y \geq 1$ and $z - x \geq 2$. Following Guillotte [4], we consider $(x/z)^i + (y/z)^i = 1 + e_i$, where $e_0 = 1$, $e_p = 0$, and $e_i \in (0, 1)$ for $1 \leq i \leq p$. Summing over i from 0 to p, Guillotte further showed that

$$1/(1 - x/z) + 1/(1 - y/z) > p + 1 + \sum_{i=0}^{p} e_i,$$

from which we obtain

$$z(1/(z - x) + 1/(z - y)) > p + 2.$$