ANOMALIES IN HIGHER-ORDER CONJUGATE QUATERNIONS: A CLARIFICATION

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1. INTRODUCTION

In a previous paper [3], brief mention was made of the conjugate quaternion $\overline{P}_n$ of the quaternion P_n. Following the definitions given by Horadam [2], Iyer [6], and Swamy [7], we have

(1) $$P_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3},$$

and consequently, its conjugate $\overline{P}_n$ is given by

(2) $$\overline{P}_n = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$$
$$jk = -kj = i, \quad ki = -ik = j.$$

In [3], T_n was defined to be a quaternion with quaternion components P_{n+r} ($r = 0, 1, 2, 3$), that is,

(3) $$T_n = P_n + iP_{n+1} + jP_{n+2} + kP_{n+3},$$

and the conjugate of T_n was defined as

(4) $$\overline{T}_n = P_n - iP_{n+1} - jP_{n+2} - kP_{n+3}$$

which, with (1), yields

(5) $$\overline{T}_n = W_n + W_{n+2} + W_{n+4} + W_{n+6}.$$

Here the matter of conjugate quaternions was laid to rest without investigating further the inconsistency that had arisen, namely, the fact that the conjugate for the quaternion T_n [defined in (4) analogously to the standard conjugate quaternion form (2)] was a scalar (5) and not a quaternion as normally defined. This inconsistency, however, made attempts to derive expressions for conjugate quaternions of higher order similar to those of higher-order quaternions established in [4] and [5], rather difficult. The change in notation from that used in [3] to the operator notation adopted in [4] and [5], added further complications. Given that $\Omega W_n \equiv P_n$ and $\Omega^2 W_n \equiv T_n$, the introduction of this operator notation created a whole new set of possible conjugates for each of the higher-order quaternions. For example, for quaternions with quaternion components (quaternions of order 2), we could apparently define the conjugate of $\Omega^2 W_n$ in several ways, viz. (6)-(9):

(6) $$\Omega\overline{\Omega}W_n = \overline{\Omega}W_n + i\overline{\Omega}W_{n+1} + j\overline{\Omega}W_{n+2} + k\overline{\Omega}W_{n+3};$$

(7) $$\overline{\Omega}\Omega W_n = \Omega W_n - i\Omega W_{n+1} - j\Omega W_{n+2} - k\Omega W_{n+3};$$

(8) $$\overline{\Omega}^2 W_n = \overline{\Omega}W_n - i\overline{\Omega}W_{n+1} - j\overline{\Omega}W_{n+2} - k\overline{\Omega}W_{n+3};$$

(9) $$\overline{\Omega^2}W_n = W_n - W_{n+2} - W_{n+4} - W_{n+6} - 2iW_{n+1} - 2jW_{n+2} - 2kW_{n+3}.$$

It is clear that the difficulties which have arisen are due, in part, to the choice of the defining notation. It is the purpose of this paper to redefine higher-order conjugate quaternions using the more descriptive nomenclature provided by the operator notation as outlined in [4]. We are thus concerned with determining the unique conjugate of a general higher-order quaternion.

2. SECOND-ORDER CONJUGATE QUATERNIONS

We begin by defining the conjugate of ΩW_n as $\overline{\Omega} W_n$ ($\equiv \overline{P}_n$, c.f. (6) in [3]), where

(10) $$\overline{\Omega} W_n = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}.$$

Consider (6) and (7) above. If we expand these expressions using (10) and (1) with $\Omega W_n = P_n$, respectively, we find that

(11) $$\Omega\overline{\Omega} W_n = \overline{\Omega}\Omega W_n = W_n + W_{n+2} + W_{n+4} + W_{n+6},$$

which is the same as (5). Since the right-hand side of (5) and (11) are independent of the quaternion vectors i, j, and k, $\Omega\overline{\Omega} W_n$, $\overline{\Omega\Omega} W_n$, and $\overline{T}_n$ are not quaternions and, therefore, cannot be defined as the conjugate of $\Omega^2 W_n$ ($= \overline{T_n}$). We emphasize that $\overline{T}_n$, as defined by (4), 9(a) of [3], is not the conjugate of T_n.

Since the expanded expression for $\Omega^2 W_n$ ($\equiv T_n$, c.f. 8(a) in [3]) is

(12) $$\Omega^2 W_n = W_n - W_{n+2} - W_{n+4} - W_{n+6} + 2iW_{n+1} + 2jW_{n+2} + 2kW_{n+3},$$

it follows that the conjugate of $\Omega^2 W_n$ must be $\overline{\Omega^2} W_n$ as given by (9). If we now take (8) and expand the right-hand side, we see that it is identical to the right-hand side of (9), so that the conjugate of $\Omega^2 W_n$ can also be denoted $\overline{\Omega}^2 W_n$.

By taking the product of $\Omega^2 W_n$ and $\overline{\Omega}^2 W_n$, we obtain

(13) $$\begin{aligned}\Omega^2 W_n \overline{\Omega}^2 W_n = {} & W_n^2 + W_{n+2}^2 + W_{n+4}^2 + W_{n+6}^2 \\ & + 4W_{n+1}^2 + 4W_{n+2}^2 + 4W_{n+3}^2 \\ & - 2W_n W_{n+2} - 2W_n W_{n+4} - 2W_n W_{n+6} \\ & + 2W_{n+2}W_{n+4} + 2W_{n+2}W_{n+6} + 2W_{n+4}W_{n+6},\end{aligned}$$

and we observe that the right-hand side of this equation is a scalar. Thus $\overline{\Omega}^2 W_n$ preserves the basic property of a conjugate quaternion.

We note in passing that as $\overline{P}_n \equiv \overline{\Omega} W_n$, the conjugate quaternion $\overline{T}_n$ should have been defined as [c.f. (8)],

(14) $$\overline{T}_n = \overline{P}_n - i\overline{P}_{n+1} - j\overline{P}_{n+2} - k\overline{P}_{n+3}.$$

3. THE GENERAL CASE

In Section 2 above, the conjugate $\overline{\Omega^2} W_n$ of $\Omega^2 W_n$ was determined by expanding the quaternion $\Omega^2 W_n$ and conjugating in the usual way. It was established that $\overline{\Omega}^2 W_n \equiv \overline{\Omega^2} W_n$. We now seek to prove that this relationship is generally true, i.e., for any integer λ, $\overline{\Omega}^\lambda W_n \equiv \overline{\Omega^\lambda} W_n$.

First, we need to derive a Binet form for the generalized conjugate quaternion of arbitrary order.

As in [5], we introduce the extended Binet form for the generalized quaternion of order λ:

(15) $$\Omega^\lambda W_n = A\alpha^n \underline{\alpha}^\lambda - B\beta^n \underline{\beta}^\lambda \quad (A, B \text{ constants})$$

where α and β are defined as in Horadam [1] and

(16) $$\begin{cases} \underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3 \\ \underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3. \end{cases}$$

We now define the conjugates $\overline{\underline{\alpha}}$ and $\overline{\underline{\beta}}$ so that

(17) $$\begin{cases} \overline{\underline{\alpha}} = 1 - i\alpha - j\alpha^2 - k\alpha^3 \\ \overline{\underline{\beta}} = 1 - i\beta - j\beta^2 - k\beta^3. \end{cases}$$

Substituting the Binet forms, as given by (1.6) in Horadam [1], for the terms on the right-hand side of (10), we obtain

$$\begin{aligned}\overline{\Omega}W_n &= A\alpha^n - B\beta^n - i(A\alpha^{n+1} - B\beta^{n+1}) - j(A\alpha^{n+2} - B\beta^{n+2}) - k(A\alpha^{n+3} - B\beta^{n+3})\\ &= A\alpha^n(1 - i\alpha - j\alpha^2 - k\alpha^3) - B\beta^n(1 - i\beta - j\beta^2 - k\beta^3),\end{aligned}$$

i.e.,

(18) $$\overline{\Omega}W_n = A\alpha^n\overline{\underline{\alpha}} - B\beta^n\overline{\underline{\beta}},$$

which is the Binet form for the conjugate quaternion $\overline{\Omega}W_n$. This result can easily be generalized by induction, so that, for λ an integer,

(19) $$\overline{\Omega}^{\lambda}W_n = A\alpha^n\overline{\underline{\alpha}}^{\lambda} - B\beta^n\overline{\underline{\beta}}^{\lambda}.$$

Lemma: For some integer λ,

$$\overline{\underline{\alpha}}^{\lambda} = \overline{\underline{\alpha}^{\lambda}} \quad \text{and} \quad \overline{\underline{\beta}}^{\lambda} = \overline{\underline{\beta}^{\lambda}}.$$

Proof: We will prove only the result for the quaternion $\underline{\alpha}$, as the proof of the result for $\underline{\beta}$ is identical. From (16) above, it follows that

(20) $$\begin{cases}\underline{\alpha}^2 &= (1 + i\alpha + j\alpha^2 + k\alpha^3)^2\\ &= 1 - \alpha^2 - \alpha^4 - \alpha^6 + 2i\alpha + 2j\alpha^2 + 2k\alpha^3\\ &= 2\underline{\alpha} - (1 + \alpha^2 + \alpha^4 + \alpha^6).\end{cases}$$

Letting

(21) $$S_\alpha = 1 + \alpha^2 + \alpha^4 + \alpha^6,$$

we have

(22) $$\underline{\alpha}^2 = 2\underline{\alpha} - S_\alpha.$$

Hence, on multiplying both sides of this equation by $\underline{\alpha}$, we obtain

$$\underline{\alpha}^3 = 2\underline{\alpha}^2 - \underline{\alpha}S_\alpha,$$

which, by (22) becomes

$$\underline{\alpha}^3 = (4 - S_\alpha)\underline{\alpha} - 2S_\alpha.$$

If we continue this process, a pattern is discernible from which we derive a general expression for $\underline{\alpha}^{\lambda}$ given by

(23) $$\begin{aligned}\underline{\alpha}^{\lambda} = &\left\{\sum_{r=0}^{\left[\frac{\lambda-1}{2}\right]}\binom{\lambda - 1 - r}{r}2^{\lambda-1-2r}(S_\alpha)^r(-1)^r\right\}\underline{\alpha}\\ &- \left\{\sum_{r=0}^{\left[\frac{\lambda-2}{2}\right]}\binom{\lambda - 2 - r}{r}2^{\lambda-2-2r}(S_\alpha)^r(-1)^r\right\}S_\alpha,\end{aligned}$$

where $\left[\frac{\lambda - 1}{2}\right]$ refers to the integer part of $\frac{\lambda - 1}{2}$.

From equations (20) and (22), it is evident that

(24) $$\overline{\underline{\alpha}^2} = 2\overline{\underline{\alpha}} - S_\alpha.$$

Since S_α is a scalar, and the only quaternion in the right-hand side of (23) is $\underline{\alpha}$, it follows that the conjugate $\overline{\underline{\alpha}^{\lambda}}$ must be

(25)
$$\overline{\underline{\alpha}^{\lambda}} = \left\{\sum_{r=0}^{\left[\frac{\lambda-1}{2}\right]}\binom{\lambda-1-r}{r}2^{\lambda-1-2r}(S_\alpha)^r(-1)^r\right\}\overline{\underline{\alpha}} - \left\{\sum_{r=0}^{\left[\frac{\lambda-2}{2}\right]}\binom{\lambda-2-r}{r}2^{\lambda-2-2r}(S_\alpha)^r(-1)^r\right\}S_\alpha.$$

We now employ the same procedure as we used above to obtain a general expression for $\underline{\alpha}^{\lambda}$ [c.f. (23)] to secure a similar result for $\overline{\underline{\alpha}}^{\lambda}$.

From (17) it ensues that

$$\overline{\underline{\alpha}}^2 = (1 - i\alpha - j\alpha^2 - k\alpha^3)^2$$
$$= 1 - \alpha^2 - \alpha^4 - \alpha^6 - 2i\alpha - 2j\alpha^2 - 2k\alpha^3,$$

i.e.,

(26)
$$\overline{\underline{\alpha}}^2 = 2\overline{\underline{\alpha}} - S_\alpha,$$

and we note that this equation is identical to (24). Multiplying both sides of (26) by $\overline{\underline{\alpha}}$ gives us

$$\overline{\underline{\alpha}}^3 = 2\overline{\underline{\alpha}}^2 - \overline{\underline{\alpha}}S_\alpha,$$

which, by (26), yields

$$\overline{\underline{\alpha}}^3 = (4 - S_\alpha)\overline{\underline{\alpha}} - 2S_\alpha.$$

It is obvious from the emerging pattern that, by repeated multiplication of both sides by $\overline{\underline{\alpha}}$ and subsequent substitution for $\overline{\underline{\alpha}}^2$ by the right-hand side of (26), the expression derived for $\overline{\underline{\alpha}}^{\lambda}$ will be precisely (25). Hence, $\overline{\underline{\alpha}}^{\lambda} = \overline{\underline{\alpha}^{\lambda}}$. Similarly, it can be shown that $\overline{\underline{\beta}}^{\lambda} = \overline{\underline{\beta}^{\lambda}}$.

Theorem: For λ an integer,
$$\overline{\Omega}^{\lambda}W = \overline{\Omega^{\lambda}W}.$$

Proof: Taking the conjugate of both sides of (15) gives us

$$\begin{aligned}\overline{\Omega^{\lambda}W_n} &= \overline{A\alpha^n\underline{\alpha}^{\lambda} - B\beta^n\underline{\beta}^{\lambda}}\\ &= A\alpha^n\overline{\underline{\alpha}^{\lambda}} - B\beta^n\overline{\underline{\beta}^{\lambda}}\\ &= A\alpha^n\overline{\underline{\alpha}}^{\lambda} - B\beta^n\overline{\underline{\beta}}^{\lambda} && \text{(Lemma)}\\ &= \overline{\Omega}^{\lambda}W_n && \text{[c.f. (19)]}\end{aligned}$$

as desired.

We have thus established that the conjugate for a generalized quaternion of order λ can be determined by taking λ operations on the conjugate quaternion operator $\overline{\Omega}$. This provides us with a rather simple method of finding the conjugate of a higher-order quaternion.

Finally, let us again consider the conjugate quaternion $\overline{\Omega}W_n$. It readily follows from (10) that

$$\overline{\Omega}W_n = 2W_n - W_n - iW_{n+1} - jW_{n+2} - kW_{n+3},$$

i.e.,

$$\overline{\Omega}W_n = 2W_n - \Omega W_n.$$

This equation relates the conjugate quaternion $\overline{\Omega}W_n$ to the quaternion ΩW_n. If we rewrite (27) as

$$\overline{\Omega}W_n = (2 - \Omega)W_n.$$

it is possible to manipulate the operators in the ensuing fashion:

$$\overline{\Omega}^2 W_n = (2 - \Omega)^2 W_n = (4 - 4\Omega + \Omega^2) W_n = 4W_n - 4\Omega W_n + \Omega^2 W_n.$$

This result can be verified directly through substitution by (1), (9), and (12), recalling that $P_n = \Omega W_n$ and $\overline{\Omega}^2 W_n = \overline{\Omega^2} W_n$. Once again, by induction on λ, it is easily shown that

$$\overline{\Omega}^\lambda W_n = (2 - \Omega)^\lambda W_n. \tag{28}$$

It remains open to conjecture whether an examination of various permutations of the operators Ω and $\overline{\Omega}$, together with the operator Δ (defined in [4]) and its conjugate $\overline{\Delta}$, will lead to further interesting relationships for higher-order quaternions.

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ON THE CONVERGENCE OF ITERATED EXPONENTIATION—II*

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In a previous paper [1], we have discussed the properties of the function $f(x)$ defined as:

$$f(x) = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}} \tag{1}$$

and a generalization of $f(x)$, namely [2, 3],

$$F_n(x) = g_1(x)^{g_2(x)^{g_3(x)^{\cdot^{\cdot^{\cdot^{g_n(x)}}}}}} = \mathop{\Xi}_{j=1}^{n} g_j(x), \tag{2}$$

where the $g_j(x)$ are functions of a positive real variable x, and the symbol Ξ is used to denote the iterated exponentiation [4]. For both (1) and (2), the

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