$$= \frac{2(2x+1)}{2x(2x+2)(2x+4)}\left\{(-1)^n[2(n+1)(x+1)-1]\binom{x+1}{n+1}\Big/\binom{2x+2}{2n+2}\right.$$

$$\left. + (-1)^a[2a(x+1)-1]\binom{x+1}{a}\Big/\binom{2x+2}{2a}\right\} \qquad \text{Q.E.D.}$$

Proof of Identity 10:

$$\sum_{k=a}^{n}(-1)^k k\binom{x}{k}\Big/\binom{2x+1}{2k} = (x+1)\sum_{k=a}^{n}(-1)^k\binom{x}{k-1}\Big/\binom{2x+2}{2k}$$

$$= \frac{2x+3}{(2x+4)(2x+6)}\left\{(-1)^n[2(n+1)(x+2)-1]\right.$$

$$\binom{x+2}{n+1}\Big/\binom{2x+4}{2n+2}$$

$$\left. + (-1)^a[2a(x+2)-1]\binom{x+2}{a}\Big/\binom{2x+4}{2a}\right\} \text{ by Identity 9}$$

Q.E.D.

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FIBONACCI CUBATURE

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Korobov [1] developed procedures for integration over an N-dimensional cube which are referred to in the literature [2, 3, 4] as number-theoretical methods or the method of optimal coefficients. These methods involve summation over a lattice of nodes defined by a single index instead of N nested summations. For the two-dimensional case, a particularly simple form involving the Fibonacci numbers is obtained. Designating the Nth Fibonacci number by F_N, k/F_N by x_k, and $\{F_{N-1}x_k\}$ by y_k, where $\{\ \}$ denotes the fractional part, the cubature rule is

$$\int_0^1\int_0^1 f(x, y)\,dxdy = \frac{1}{F_N}\sum_{k=1}^{F_N} f(x_k, y_k). \tag{1}$$

The summation can also be taken as running from 0 to $F_N - 1$, which replaces a node 1, 0 by 0, 0 while leaving the rest unchanged. This cubature rule was also given by Zaremba [5].

The investigators have been interested primarily in the higher-dimensional cases and very little has been published on the two-dimensional case. An examination of the nodes for the two-dimensional case suggested an interesting conjecture about their symmetry properties and a modification which improves the accuracy significantly.

Conjecture: If x_k, y_k is a node for $1 \leq k \leq F_N - 1$ and if N is $\binom{\text{odd}}{\text{even}}$, then $\binom{y_k,\ 1 - x_k}{y_k,\ x_k}$ is also a node.

Perhaps a reader can supply a proof.

One would expect the nodes of an efficient cubature rule to be symmetric about the center of the square so as to give identical results for $f(x, y)$, $f(x, 1 - y)$, $f(1 - x, y)$, and $f(1 - x, 1 - y)$. This suggests modifying (1) to

$$\int_0^1 \int_0^1 f(x, y)\, dx dy = \frac{f(0, 0) + f(0, 1) + \sum_{k=1}^{F_N} f(x_k, y_k) + f(x_k, 1 - y_k)}{2(F_N + 1)}. \tag{2}$$

Essentially, we have completed the square on the nodes. Some preliminary calculations* indicated that this gain in accuracy more than compensated for doubling the number of function evaluations.

The performance of the method is reasonably good, although it is not competitive with a high-order-product Gauss rule using a comparable number of nodes. It might be a useful alternative for use on programmable hand calculators which do not have the memory to store tables of weights and nodes and where the use of only one loop in the algorithm is a significant advantage.

I also plan to investigate the effect of the symmetrization in higher-dimensional calculations, but in such cases the number of nodes increases very rapidly with the dimensionality.

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*I am indebted to Mr. Robert Harper, a graduate student in the Department of Chemical Engineering for programming the procedure on a T159.

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ON A PROBLEM OF S. J. BEZUSZKA AND M. J. KENNEY ON CYCLIC DIFFERENCE OF PAIRS OF INTEGERS

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Begin with four nonnegative integers, for example, a, b, c, and d. Take cyclic difference of pairs of integers (the smaller integer from the larger), where the fourth difference is always the difference between the last integer