$$\sum_{x \le i \le y} R_i S_i = \left[\frac{pq^2(R_n S_{n+1} + R_{n+1}S_n) + (1 - q)[R_{n+2}S_{n+2} + (1 - p^2)R_{n+1}S_{n+1}]}{(1 - q)(p + q - 1)(p - q + 1)}\right]_{n=x-1}^{n=y},$$

$$\text{if } q + 1 \ne 0,\ p + q - 1 \ne 0,\ p - q + 1 \ne 0. \tag{16}$$

In closing, we note that the expressions of this paper can be used to derive some identities among recurrence terms. As an example consider $\Sigma R_i S_i$ with R_i and S_i identical sequences, $R_0 = S_0 = 0$, $R_1 = S_1 = 1$, $p = 1$, $q = 2 + \varepsilon$, and limits of summation $0 \le i \le n$. As $\varepsilon \to 0$, the sum approaches a well-defined value, and thus the right-hand side of (16) must also have a finite limit. Since the denominator goes to 0, so must the numerator. We conclude that the following must be true:

$$\left[8R_y R_{y+1} - R_{y+2}^2\right]_{y=-1}^{y=n} = 8R_n R_{n+1} - R_{n+2}^2 + 1 = 0$$

or

$$8R_n R_{n+1} = (R_{n+2} + 1)(R_{n+2} - 1)$$

$$\text{if } p = 1,\ q = 2,\ R_0 = 0,\ R_1 = 1.$$

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ITERATING THE PRODUCT OF SHIFTED DIGITS

SAMUEL S. WAGSTAFF, JR.
Northern Illinois University, DeKalb, IL 60115

1. INTRODUCTION

Let t be a fixed nonnegative integer. For positive integers n written in decimal as

$$n = \sum_{i=0}^{k} d_i \cdot 10^i,$$

with $0 \le d_i \le 9$ and $d_k > 0$, we define

$$f_t(n) = \prod_{i=0}^{k} (t + d_i).$$

Also define $f_0(0) = 0$. Erdös and Kiss [1] have asked about the behavior of the sequence of iterates n, $f_t(n)$, $f_t(f_t(n))$, ... They noted that $f_4(120) = 120$. For $t = 0$, every such sequence eventually reaches a one-digit number. Sloane

[2] has considered this case. For $t = 1$, we prove that the sequence of iterates from any starting point n remains bounded, and we list the two possible cycles. For $t \geq 10$, it is clear that $f_t(n) > n$ for every n so that the sequence always tends to infinity. We discuss the cases $2 \leq t \leq 9$ and present numerical evidence and a heuristic argument which conclude that every sequence remains bounded when $t \leq 6$, while virtually every sequence tends to infinity for $t \geq 7$. In Table 1 we give the known cycles in which these sequences may be trapped when $0 \leq t \leq 6$. See also [3] for the case $t = 0$.

TABLE 1. Some Data on the Cycles of f_t for $0 \leq t \leq 6$

t	Least Term of Cycle	Cycle Length	First Start Leading to it	# of Starts ≤ 100000 Leading to it
0	0	1	10	82402
	1	1	1	5
	2	1	2	3213
	3	1	3	15
	4	1	4	894
	5	1	5	607
	6	1	6	6843
	7	1	7	15
	8	1	8	5971
	9	1	9	35
1	2	9	1	92043
	18	1	18	7957
2	6	3	2	9927
	9	2	1	6
	12	1	12	29105
	24	1	16	60105
	35	1	35	2
	56	1	56	811
3	24	10	1	47955
	648	2	134	52045
4	96	5	1	6793
	112	16	37	70677
	120	1	29	20
	315	1	135	6
	1280	2	589	4798
	2688	3	1289	6971
	4752	1	1157	90
	7744	1	4477	185
	15840	2	4779	9992
	24960	1	10489	378
	57915	1	15579	90
5	50	1	50	1
	210	1	57	6
	450	1	3	222
	780	1	158	10
	1500	1	4	35726

(continued)

TABLE 1 (continued)

t	Least Term of Cycle	Cycle Length	First Start Leading to it	# of Starts $\leq$ 100000 Leading to it
5	1600	3	228	7058
	3920	1	22	91
	16500	1	1339	146
	16800	4	1	4927
	32760	4	368	51483
	91728	1	11899	300
	1293600	1	38899	30
6	90	1	34	3
	840	1	4	40
	4320	1	3	329
	9360	2	35	550
	51744	5	18	2626
	59400	1	7899	300
	60480	1	6	3300
	917280	1	7777	493
	2419200	1	26778	12
	533744640	62	38	10968
	1556755200	21	1	25484
	139089000960	85	5	5895

2. THE CASE $t = 1$

This is the only nontrivial case in which we can prove that every sequence of iterates is bounded.

Theorem: Let n be a positive integer. Then $f_1(n) = n$ if and only if $n = 18$. Also $f_1(n) > n$ if and only if $n = d \cdot 10^k - 1$, where $k \geq 0$ and $2 \leq d \leq 10$. In In the latter case, $f_1(n) = n + 1$. Iteration of f_1 from a positive starting number eventually leads either to the fixed point 18 or to the cycle (2, 3, 4, 5, 6, 7, 8, 9, 10).

Proof: If $n = d \cdot 10^k - 1$ with $2 \leq d \leq 10$, then the digits of n are $d - 1$ and k nines. Thus $f_1(n) = d \cdot 10^k = n + 1$. Now suppose $k \geq 1$ and n has $k + 1$ digits, but n is not of the form $d \cdot 10^k - 1$. Then the low-order k digits are not all nines. Write

$$n = \sum_{i=0}^{k} d_i \cdot 10^i,$$

and let j be the greatest subscript such that $j < k$ and $d_j < 9$. Then

$$\begin{aligned} f_1(n) &\leq (d_k + 1)(d_j + 1)10^{k-1} \\ &= d_k \cdot 10^k + d_j \cdot 10^{k-1} + (1 + d_k(d_j - 9))10^{k-1}. \end{aligned} \tag{1}$$

Now $d_j - 9 \leq -1$ and $d_k \geq 1$. Hence the last term of (1) is nonpositive, and it vanishes if and only if $d_k = 1$ and $d_j = 8$. Hence $f_1(n) < n$ if either $d_k > 1$ or $d_j < 8$. If $d_k = 1$ and $d_j = 8$ and $j < k - 1$, then also $f_1(n) < n$. Otherwise, either $n = 18$ [and $f_1(18) = 18$] or n has at least three digits, the first two of which are 18. If any lower-order digit were nonzero, the inequality in (1) would be strict and give $f_1(n) < n$. Finally, if $n = 1800...0$, clearly

$$f_1(n) = (1 + 1)(8 + 1) = 18 < n.$$

The last statement of the theorem follows easily from the earlier ones by induction on n. For $n > 18$, either $f_1(n) < n$ or $f_1(f_1(n)) < n$.

3. THE CASES $t = 2$ THROUGH 6

These five cases are alike in that there is compelling evidence that all the sequences are bounded, but we cannot prove it. In Table 1 we gave some data on the known cycles of f for $0 \leq t \leq 6$. Table 2 lists the cycles of length > 1. For $t \leq 5$, every starting number up to 100000 eventually reaches one of these cycles. For $t = 6$, the same is true up to 50000.

TABLE 2. Cycles of at Least Two Terms

t	Cycle
1	(2, 3, 4, 5, 6, 7, 8, 9, 10)
2	(6, 8, 10)
2	(9, 11)
3	(24, 35, 48, 77, 100, 36, 54, 56, 72, 50)
3	(648, 693)
4	(96, 130, 140, 160, 200)
4	(112, 150, 180, 240, 192, 390, 364, 560, 360, 280, 288, 864, 960, 520, 216, 300)
4	(1280, 1440)
4	(2688, 8640, 3840)
4	(15840, 17280)
5	(1600, 1650, 3300)
5	(16800, 21450, 18900, 27300)
5	(32760, 36960, 67760, 87120)
6	(9360, 9720)
6	(51744, 100100, 63504, 71280, 61152)
6	(533744640, 833976000, 573168960, 1634592960, 10777536000, 23678246592, 199264665600, 1034643456000, 1163973888000, 5504714691840, 6992425440000, 2463436800000, 1015831756800, 2466927695232, 20495794176000, 36428071680000, 14379662868480, 27960455776000, 6548726486016000, 7030057402368000, 9442156179456O, 11987015040000O, 2883421981440O, 4182119424000O, 5974456320000O, 264203596800O, 248314429440O, 304819200000O, 29628426240O, 44590694400O, 38491200000O, 4938071040O, 2228990400O, 2090188800O, 17923368960, 160487308800, 349505694720, 1100848320000, 322620641280, 187280916480, 906125875200, 383584481280, 115008284160O, 920066273280, 391283343360, 499979692800, 477640800000O, 794794291200, 919900800000, 92588832000, 56330588160, 69709102848, 138692736000, 385169541120, 451818259200, 401616230400, 65840947200, 62270208000, 8695185408, 25101014400, 3911846400, 4000752000)

TABLE 2 (continued)

t	Cycle
6	(1556755200, 4604535936, 12702096000, 8151736320, 4576860288, 27122135040, 11623772160, 28848089088, 325275955200, 473609410560, 420323904000, 60466176000, 24455208960, 70253568000, 24659002368, 68976230400, 61138022400, 10241925120, 10431590400, 9430344000, 1574640000)
	(139089000960, 277766496000, 984031027200, 142655385600, 486857226240, 1239869030400, 2222131968000, 983224811520, 438126796800, 998587699200, 4903778880000, 4868115033600, 2661620290560, 2648687247360, 19781546803200, 38445626419200, 48283361280000, 15485790781440, 106051785840000, 84580378122240, 45565186867200, 118144020234240, 47795650560000, 37781114342400, 18931558464000, 40663643328000, 18284971622400, 41422897152000, 16273281024000, 6390961274880, 14978815488000, 87214615488000, 39869538508800, 219583673971200, 642591184435200, 309818234880000, 203251004006400, 14898865766400, 256304176128000, 105450861035520, 112464019261440, 119489126400000, 80655160320000, 5736063320064, 3112798740480, 6310519488000, 2218016908800, 2007417323520, 1165698293760, 1647669703680, 100144080691200, 32262064128000, 6742112993280, 6657251328000, 2761808265216, 7290429898752, 37777259520000, 38697020144640, 42796615680000, 37661021798400, 38944920268800, 92177326080000, 13352544092160, 19916886528000, 82805964595200, 97371445248000, 42499416960000, 35271936000000, 5447397795840, 45218873700000, 14279804098560, 91537205760000, 14425516385280, 53013342412800, 7604629401600, 2445520896000, 2529128448000, 2503581696000, 2390026383360, 2742745743360, 9020284416000, 877879296000, 2009063347200, 943272345600, 480370176000)

Some cycles may by reached from only finitely many starting numbers. For example, it is easy to see that $f_5(n) = 50$ only when $n = 50$. The cycle (9, 11) for f_2 may be reached only from the odd numbers below 12. Only 35 and 53 lead to the fixed point 35 of f_2. It is a ten-minute exercise to discover all twenty starting numbers which lead to the fixed point 120 of f_4. The fixed point 90 for f_6 may be reached only from the starting numbers 34, 43, and 90.

Given a cycle, what is the asymptotic density of the set of starting numbers which lead to it? We cannot answer this question even for the two cycles for $t = 1$. Some relevant numerical data is shown in the last column of Table 1. Since the digit 0 occurs in almost all numbers, the answer to the question is clear in case $t = 0$.

4. THE CASES $t = 7$ THROUGH 9

The starting number 5 leads to the fixed point 31746120037632000 of f_7. We found no other cycles in these three cases. Every sequence with starting number up to 1000 rises above 10^{14}. Every sequence starting below 17 (except 5 and 12 for f_7) rises above 10^{300}. These observations, together with the heuristic argument below, suggest that nearly every sequence diverges to infinity.

When $7 \leq t \leq 9$, it usually happens that $f_t(n) > n$. The least n with $f_t(n) \leq n$ is 700, 9000, 90000000, for $t = 7, 8, 9$, respectively.

The sequences show a strong tendency to merge. We conjecture that there is a finite number of sequences such that every sequence merges with one of them.

5. THE HEURISTIC ARGUMENT

Let t be a fixed positive integer. Consider a positive number n of k digits, where k is large. For $0 \leq d \leq 9$ and most n, about $k/10$ of the digits will be d. Thus

$$f_t(n) \approx \prod_{d=0}^{9} (d + t)^{k/10} = (p_t)^k, \text{ where } p_t = \left(\prod_{d=0}^{9} (d + t)\right)^{1/10}.$$

This means that $f_t(n)$ will have about $k \cdot \log_{10} p_t$ digits. From Table 3, it is clear that this implies that $f_t(n) < n$ for most large n when $1 \leq t \leq 5$, and that $f_t(n) > n$ for most large n when $6 \leq t \leq 9$.

It is tempting to apply the same reasoning to the subsequent terms of the sequence. Note, however, that $f_t(n)$ cannot be just any number. About one-fifth of the digits of n are $\equiv -t \pmod 5$ and about half of them have the same parity as t. Hence the highest power of 10 that divides $f_t(n)$ is usually about $10^{k/5}$, so that $f_t(n)$ will have many more zero digits than other numbers of comparable size. It is plausible that, after several iterations, the fraction of digits which are low-order zeros will stabilize. Furthermore, it is likely that the significant digits will take on the ten possible values with equal frequency. Suppose we reach a number m of k digits. Assume there are constants a, b, s, which depend on t but not on m or k, so that (i) m has about ak low-order zeros, (ii) each of the ten digits occurs about bk times as a significant digit of m, and (iii) $f_t(m)$ has about sk digits, of which approximately ask are low-order zeros. Then $a + 10b = 1$ and

$$ask \approx \min(\mathrm{ord}_2(f_t(m)), \mathrm{ord}_5(f_t(m))), \tag{2}$$

where $\mathrm{ord}_p(w)$ denotes the ordinal of w at the prime p. By hypotheses (i) and (ii), we have

$$f_t(m) \approx (0 + t)^{ak+bk}(1 + t)^{bk} \ldots (9 + t)^{bk} = (t^a \pi_t^b)^k,$$

where

$$\pi_t = \prod_{d=0}^{9} (d + t).$$

Since $sk \approx \log_{10} f_t(m)$, we find

$$s \approx a \log_{10} t + b \log_{10} \pi_t. \tag{3}$$

When $t = 5$, equation (2) becomes

$$ask \approx \min(8bk, ak + 2bk)$$

because $8 = \mathrm{ord}_2 \pi_5$. Hence $as \approx 8b$, so

$$a \approx \frac{8}{8 + 10s} \quad \text{and} \quad b \approx \frac{s}{8 + 10s}.$$

Substitution in (3) gives a quadratic equation in s whose positive root is shown in Table 3, together with a and b.

If $1 \leq t \leq 9$ and $t \neq 5$, then (2) becomes

$$ask \approx \min(gak + hbk, bk + bk),$$

where $g \geq 0$ and $h \geq 7$. Hence $as \approx 2b$, and we find

$$a \approx \frac{2}{2 + 10s} \quad \text{and} \quad b \approx \frac{s}{2 + 10s}.$$

Using (3) produces a quadratic equation in s whose positive root is given in Table 3.

TABLE 3. Values of p_t and s for $1 \le t \le 9$

t	p_t	$\log_{10} p_t$	a	b	s
1	4.5	0.66	0.30	0.070	0.46
2	5.8	0.76	0.23	0.077	0.65
3	6.9	0.84	0.21	0.079	0.76
4	8.0	0.90	0.19	0.081	0.84
5	9.0	0.96	0.49	0.051	0.83
6	10.086	1.0037	0.17	0.083	0.965
7	11.1	1.05	0.16	0.084	1.013
8	12.2	1.08	0.16	0.084	1.06
9	13.2	1.12	0.15	0.085	1.09

We may defend the third hypothesis this way: If we had assumed that $f_t(m)$ had about the same number of digits as m, i.e., that $s = 1$, and followed the remainder of the argument above, we would have concluded that the sequence forms an approximate geometric progression, which is the essence of (iii). There is no other simple assumption for the change in the number of digits from one term to the next.

The few sequences we studied with $7 \le t \le 9$ behaved roughly in accordance with the three hypotheses and the data in Table 3.

In summary, for most large n, $f_t(n)$ will have many fewer digits than n for $1 \le t \le 5$, about 0.37% more digits when $t = 6$, and substantially more digits for $7 \le t \le 9$. However, after several iterations, when we reach a number m, say, it will usually happen that $f_t(m)$ has many fewer digits than m for $1 \le t \le 6$ and many more digits for $7 \le t \le 9$. Thus if we iterate f_t, the sequence almost certainly will diverge swiftly to infinity for $7 \le t \le 9$, but remain bounded for $1 \le t \le 6$.

Numbers in the image of f_t not only are divisible by a high power of 10, but all their prime factors are below $10 + t$. How this property affects the distribution of digits in such numbers is unclear. There are only $O(\log^r x)$ of them up to x, where r is the number of primes up to $9 + t$.

Let $1 \le t \le 6$, and suppose that iteration of f_t from any starting number does lead to a cycle. How many iterations will be required to reach the cycle? The above heuristic argument predicts that about

$$(\log_{10}\log_{10} n)/(-\log_{10} s) + O(1)$$

iterations will be needed, which is very swift convergence indeed. In the case $t = 0$, Sloane [2] has conjectured that a one-digit number will be reached in a bounded number of iterations. The sequence for $t = 6$ starting at $n = 5$ does not enter the 85-term cycle until the 121st iterate.

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ON MAXIMIZING FUNCTIONS BY FIBONACCI SEARCH

REFAEL HASSIN
Department of Statistics, Tel Aviv University, Tel Aviv 69978

1. INTRODUCTION

The search for a local maximum of a function $f(x)$ involves a sequence of function evaluations, i.e., observations of the value of $f(x)$ for a fixed value of x. A sequential search scheme allows us to evaluate the function at different points, one after the other, using information from earlier evaluations to decide where to locate the next ones. At each stage, the smallest interval in which a maximum point of the function is known to lie is called the *interval of uncertainty*.

Most of the theoretical search procedures terminate the search when either the interval of uncertainty is reduced to a specific size or two successive estimates of the maximum are closer than some predetermined value. However, an additional termination rule which surprisingly has not received much attention by theorists exists in most practical search codes, namely the number of function evaluations cannot exceed a predetermined number, which we denote by N.

A well-known procedure designed for a fixed number of function evaluations is the so-called Fibonacci search method. This method can be applied whenever the function is unimodal and the initial interval of undertainty is finite. In this paper, we propose a two-stage procedure which can be used whenever these requirements do not hold. In the first stage, the procedure tries to bracket the maximum point in a finite interval, and in the second it reduces this interval using the Fibonacci search method or a variation of it developed by Witzgall.

2. THE BRACKETING ALGORITHM

A function f is *unimodal* on $[a, b]$ if there exists $a \leq \bar{x} \leq b$ such that $f(x)$ is strictly increasing for $a \leq x < \bar{x}$ and strictly decreasing for $\bar{x} < x \leq b$. It has been shown (Avriel and Wilde [2], Kiefer [6]) that the Fibonacci search method guarantees the smallest final interval of uncertainty among all methods requiring a fixed number of function evaluations. This method and its variations (Avriel and Wilde [3], Beamer and Wilde [4], Kiefer [6], Oliver and Wilde [7], Witzgall [10]) use the following idea:

Suppose y and z are two points in $[a, b]$ such that $y < z$, and f is unimodal, then

$$\begin{aligned} f(y) < f(z) &\quad \text{implies} \quad y \leq \bar{x} \leq b, \\ f(y) > f(z) &\quad \text{implies} \quad a \leq \bar{x} \leq z, \text{ and} \\ f(y) = f(z) &\quad \text{implies} \quad y \leq \bar{x} \leq z. \end{aligned}$$

Thus the property of unimodality makes it possible to obtain, after examining $f(y)$ and $f(z)$, a smaller new interval of uncertainty. When it cannot be said in advance that f is unimodel, a similar idea can be used.

Suppose that $f(x_1)$, $f(x_2)$, and $f(x_3)$ are known such that

$$x_1 < x_2 < x_3 \quad \text{and} \quad f(x_2) \geq \max\{f(x_1), f(x_3)\}, \tag{1}$$