ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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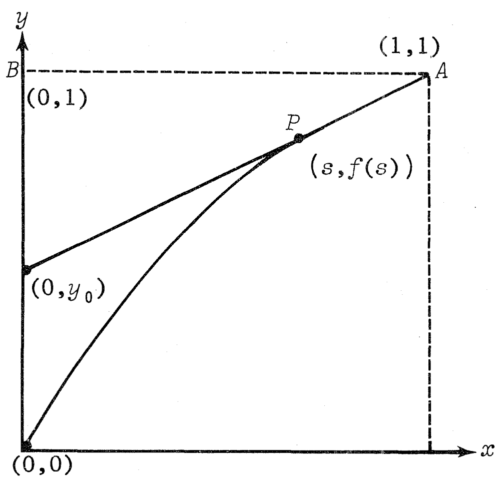
Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. Preference will be given to solutions that are submitted on separate, signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-333 *Proposed by Paul S. Bruckman, Concord CA*

The following problem was suggested by Problem 307 of *536 Puzzles & Curious Problems*, by Ernest Dudeney, edited by Martin Gardner (New York: Charles Scribner's Sons, 1967).



Leonardo and the pig he wishes to catch are at points A and B, respectively, one unit apart (which we may consider some convenient distance, e.g., 100 yards). The pig runs straight for the gateway at the origin, at uniform speed. Leonardo, on the other hand, goes directly toward the pig at all times, also at a uniform speed, thus taking a curved course. What must be the ratio r of Leonardo's speed to the pig's, so that Leonardo may catch the pig just as they both reach the gate?

H-334 *Proposed by Lawrence Somer, Washington, D.C.*

Let the Fibonacci-like sequence $\{H_n\}_{n=0}^{\infty}$ be defined by the relation

$$H_{n+2} = aH_{n+1} + bH_n,$$

where a and b are integers, $(a, b) = 1$, and $H_0 = 0$, $H_1 = 1$. Show that if p is an odd prime such that $-b$ is a quadratic nonresidue of p, then $p \nmid H_{2n+1}$ for any $n \geq 0$. (This is a generalization of Problem B-224, which appeared in the Dec. 1971 issue of *The Fibonacci Quarterly*.

SOLUTIONS

Convergents

H-311 *Proposed by Paul S. Bruckman, Concord, CA*
(Vol. 18, No. 1, February 1980)

Let a and b be relatively prime positive integers such that ab is not a perfect square. Let $\theta_0 = \sqrt{b/a}$ have the continued fraction expansion

$$[u_1, u_2, u_3, \ldots],$$

with convergents p_n/q_n ($n = 1, 2, \ldots$); also, define $p_0 = 1$, $q_0 = 0$, and $p_{-1} = 0$. The process of finding the sequence $(u_n)_{n=1}^{\infty}$ may be described by the recursions:

$$(1) \qquad \theta_n = u_{n+1} + 1/\theta_{n+1} = \frac{\sqrt{ab} + r_n}{d_n}, \text{ where } r_0 = 0,\ d_0 = a,\ 0 < \theta_n < 1,$$

r_n and d_n are positive integers, $n = 1, 2, \ldots$.

Prove:

$$(2) \qquad r_n = (-1)^{n-1}(ap_n p_{n-1} - bq_n q_{n-1});$$

$$(3) \qquad d_n = (-1)^n (ap_n^2 - bq_n^2),\ n = 0, 1, 2, \ldots .$$

Solution by the proposer

Proof: Let S denote the set of nonnegative integers n for which (2) and (3) both hold. Note that

$$ap_0 p_{-1} - bq_0 q_{-1} = a \cdot 1 \cdot 0 - b \cdot 0 = 0 = r_0$$

and

$$(-1)^0 (ap_0^2 - bq_0^2) = a \cdot 1 - b \cdot 0 = a = d_0;$$

hence, $0 \in S$. Suppose $m \in S$. Then

$$1/\theta_{m+1} = \theta_m - u_{m+1} = \frac{\sqrt{ab} + r_m}{d_m} - u_{m+1} = \frac{\sqrt{ab} - (-1)^m (ap_m p_{m-1} - bq_m q_{m-1})}{(-1)^m (ap_m^2 - bq_m^2)} - u_{m+1}$$

$$= \frac{\sqrt{ab} - (-1)^m (ap_m p_{m-1} - bq_m q_{m-1} + ap_m^2 u_{m+1} - bq_m^2 u_{m+1})}{(-1)^m (ap_m^2 - bq_m^2)}$$

$$= \frac{\sqrt{ab} - (-1)^m \{ap_m (u_{m+1} p_m + p_{m-1}) - bq_m (u_{m+1} q_m + q_{m-1})\}}{(-1)^m (ap_m^2 - bq_m^2)}$$

$$= \frac{\sqrt{ab} - (-1)^m (ap_m p_{m+1} - bq_m q_{m+1})}{(-1)^m (ap_m^2 - bq_m^2)}; \text{ therefore, } \theta_{m+1}$$

$$= (-1)^m (ap_m^2 - bq_m^2) \cdot \frac{\sqrt{ab} + (-1)^m (ap_{m+1} p_m - bq_{m+1} q_m)}{ab - (ap_{m+1} p_m - bq_{m+1} q_m)^2}.$$

However, $ab - (ap_{m+1}p_m - bq_{m+1}q_m)^2$

$$= ab - (ap_m^2 - bq_m^2)(ap_{m+1}^2 - bq_{m+1}^2) - ab(p_{m+1}q_m - q_{m+1}p_m)^2$$

$$= -(ap_m^2 - bq_m^2)(ap_{m+1}^2 - bq_{m+1}^2),$$

since

$$p_{m+1}q_m - q_{m+1}p_m = (-1)^{m+1}.$$

Thus, using the inductive hypothesis,

$$\theta_{m+1} = \frac{\sqrt{ab} + (-1)^m (ap_{m+1}p_m - bq_{m+1}q_m)}{(-1)^{m+1}(ap_{m+1}^2 - bq_{m+1}^2)},$$

which is the assertion of (2) and (3) for $n = m + 1$. Hence, $m \in S \Longrightarrow (m + 1) \in S$. By induction, (2) and (3) hold for all n.

Sum Series

H-312 *Proposed by L. Carlitz, Duke University, Durham, NC (Vol. 18, No. 1, February 1980)*

Let m, r, s be nonnegative integers. Show that

$$(*) \qquad \sum_{j,k} (-1)^{j+k-r-s}\binom{j}{r}\binom{k}{s}\frac{m!}{(m-j)!(m-k)!(j+k-m)!} = (-1)^{m-r}\binom{m}{r}\delta_{rs},$$

where
$$\delta_{rs} = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s). \end{cases}$$

Solution by Paul S. Bruckman, Concord, CA

Make the following definition:

$$(1) \qquad \theta(r, s, m) \equiv \sum_{j,k} (-1)^{j+k-r-s}\binom{j}{r}\binom{k}{s}\frac{m!}{(m-j)!(m-k)!(j+k-m)!}.$$

It may be noted, by symmetry, that

$$(2) \qquad \theta(r, s, m) = \theta(s, r, m).$$

Making the substitution $j + k = u$, we then obtain:

$$\begin{aligned}
\theta(r, s, m) &= \sum_{j,u} (-1)^{u-r-s}\binom{j}{r}\binom{u-j}{s}\frac{m!}{(m-j)!(m+j-u)!(u-m)!} \\
&= \sum_{j,u} (-1)^{u-r-s}\binom{j}{r}\binom{u-j}{s}\binom{m}{j}\binom{j}{u-m} \\
&= \sum_{j} (-1)^{r+s}\binom{j}{r}\binom{m}{j}\sum_{u} (-1)^{m+j-u}\binom{m-u}{s}\binom{j}{j-u} \qquad \text{(substituting } m+j-u \text{ for } u\text{)} \\
&= (-1)^{m+r+s}\sum_{j} (-1)^{j}\binom{m}{j}\binom{j}{r}\sum_{u} (-1)^{u}\binom{m-u}{m-s-u}\binom{j}{u} \\
&= (-1)^{m+r+s}\sum_{j} (-1)^{j}\binom{m}{j}\binom{j}{r}\sum_{u}\binom{-s-1}{m-s-u}\binom{j}{u}(-1)^{m-s}
\end{aligned}$$

(using the "negative binomial" coefficient relationship).

Now employing the Vandermonde convolution formula, we find that

$$(3) \qquad \theta(r, s, m) = (-1)^r \sum_{j} (-1)^j \binom{m}{j}\binom{j}{r}\binom{j-s-1}{m-s}.$$

Since $\theta(r, s, m) = \theta(s, r, m)$, we may without loss of generality assume $r \geq s$. In (3), note that $s \leq r \leq j \leq m$. Since $j - s - 1 < m - s$, the binomial coefficient $\binom{j-s-1}{m-s}$ vanishes whenever $j - s - 1 > 0$. If $j = s+1$, $\binom{j-s-1}{m-s} = \delta_{ms}$; however, $s = m$ implies $j = s$, a contradiction, which implies that $\binom{j-s-1}{m-s} = 0$ whenever $j > s$. The only remaining possibility is $j = s$, which implies $r = s = j$.

Hence, all terms in (3) vanish except if $r = s$, in which case (3) reduces to the single term obtained by setting $j = r = s$. In this exceptional case,

$$\theta(r, r, m) = (-1)^{2r}\binom{m}{r}\binom{r}{r}\binom{-1}{m-r} = (-1)^{m-r}\binom{m}{r}.$$

Hence, if $r \geq s$,

$$\theta(r, s, m) = (-1)^{m-r}\binom{m}{r}\delta_{rs}. \tag{4}$$

Clearly, this expression is also true if $r \leq s$, by use of (2). Q.E.D.

Also solved by the proposer.

Form Partitions!

H-313 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA (Vol. 18, No. 2, April 1980)*

(A) Show that the Fibonacci numbers partition the Fibonacci numbers.
(B) Show that the Lucas numbers partition the Fibonacci numbers.

Solution by Paul Bruckman, Concord, CA

The following definition (paraphrased) is recalled from the source indicated in the statement of the problem.

Definition: If U and V are subsets of the natural numbers, U is said to *partition V into* the subsets V_1 and V_2 if there exist subsets V_1 and V_2 of V with the following properties:

(1) $V_1 \cap V_2 = \emptyset$;
(2) $V_1 \cup V_2 = V$;
(3) $x, y \in V_i$ with $x < y$ ($i = 1$ or 2) $\Longrightarrow (x + y) \notin U$.

We also say that *U partitions V uniquely into* the subsets V_1 and V_2 if it partitions V into the subsets V_1 and V_2, and if such subsets are uniquely determined.

We will have recourse to the following theorem (see [1]):

Theorem: If r, s, and t are integers with $2 \leq r < s$, $t \geq 0$, all Diophantine solutions (r, s, t) of the equations indicated below are as follows:

(4) $F_r + F_s = F_t \Longleftrightarrow (r, s, t) = (r, r + 1, r + 2)$;

(5) $F_r + F_s = L_t \Longleftrightarrow (r, s, t) = (r, r + 2, r + 1)$ *or* $(2, 3, 2)$.

Proof of A: Set

$$U = V = F \equiv (F_n)_{n=2}^{\infty},\ V_1 = P_1 \equiv (F_{2n})_{n=1}^{\infty},\ V_2 = P_2 \equiv (F_{2n+1})_{n=1}^{\infty}.$$

Clearly, P_1 and P_2 satisfy (1) and (2), since F is a strictly increasing sequence. Suppose $x, y \in P_i$, $x < y$ ($i = 1$ or 2), and $(x + y) \in F$. Then there exist unique integers r, s, t, with $2 \leq r < s$ and $t \geq 4$ (since $F_t \geq F_2 + F_3 = F_4$), such that $x = F_r$, $y = F_s$, and $x + y = F_t$. Since x and y are in the same set P_1 or P_2, thus $s - r \geq 2$. This, however, contradicts (4), which implies that $s - r = 1$. Thus, the supposed condition is impossible, and its negation must be true, i.e., (3), with the sets as designated. Hence, F partitions F into the subsets P_1 and P_2.

To show uniqueness, suppose that F partitions F into the subsets P_3 and P_4, which are distinct from P_1 and P_2. Then, there exists an integer $u \geq 2$ such that F_u, $F_{u+1} \in P_i$ ($i = 3$ or 4). This, however, would imply $F_u + F_{u+1} = F_{u+2} \in F$, contradicting (3) and the supposition. Therefore, F partitions F uniquely into the subsets P_1 and P_2. Q.E.D.

Proof of B: Set

$$U = L \equiv (L_n)_{n=0}^{\infty},\ V = F,\ V_1 = Q_1 \equiv \{F_n : n \geq 2,\ n \equiv 1 \text{ or } 2 \pmod 4\},$$

$$V_2 = Q_2 \equiv \{F_n : n \geq 3,\ n \equiv 0 \text{ or } 3 \pmod 4\}.$$

Clearly, Q_1 and Q_2 satisfy (1) and (2). Suppose $x, y \in Q_i$, $x < y$ ($i = 1$ or 2), and $(x + y) \in L$. Then there exist unique integers r, s, t with $2 \leq r < s$ and $t \geq 2$ (since $L_t \geq F_4 = 3$, as before, and L is an increasing sequence, after the first term), such that $x = F_r$, $y = F_s$, and $x + y = L_t$. A moment's reflection shows that, since x and y are in the same set Q_1 or Q_2, thus $s - r = 1$ or 3. Since $1 = F_2 \in Q_1$ and $2 = F_3 \in Q_2$, we must not include the solution (2, 3, 2) of (5). However, for the other solutions of (5), $s - r = 2$, which also excludes those solutions. Thus, the supposed condition is impossible, which implies (3), with the sets as designated. Hence, L partitions F into the subsets Q_1 and Q_2.

To show uniqueness, suppose that L partitions F into the subsets Q_3 and Q_4, which are distinct from Q_1 and Q_2. It is readily seen that, in this case, there exists an integer $u \geq 2$ such that F_u, $F_{u+2} \in Q_i$ ($i = 3$ or 4). This, however, would imply $F_u + F_{u+2} = L_{u+1} \in L$, contradicting (3) and the supposition. Therefore, L partitions F uniquely into the subsets Q_1 and Q_2. Q.E.D.

Reference: [1] Private correspondence with Professor Verner E. Hoggatt, Jr. (June 1980), in which allusion is made to *Fibonacci and Lucas Numbers* by V. E. Hoggatt, Jr. (The Fibonacci Association, 1969), p. 74, and to Problem E 1424 in *The American Mathematical Monthly* proposed by V. E. Hoggatt, Jr. The Theorem follows from Zeckendorf's Theorem.

Also solved by the proposer.

It's the Limit

H-314 *Proposed by Paul S. Bruckman, Concord, CA*
(Vol. 18, No. 2, April 1980)

Given $x_0 \in (-1, 0)$, define the sequence $S = (x_n)_{n=0}^{\infty}$ as follows:

(1) $$x_{n+1} = 1 + (-1)^n \sqrt{1 + x_n},\ n = 0, 1, 2, \ldots .$$

Find the limit point(s) of S, if any.

Solution by the proposer

We will show that S has precisely two limit points and that

(2) $$x_{2n} \to \beta \quad \text{and} \quad x_{2n+1} \to \alpha,$$

where α and β are the Fibonacci constants. We first prove, by induction, that

(3) $$-1 < x_{2n} < 0,\ 1 < x_{2n+1} < 2,\ n = 0, 1, 2, \ldots .$$

Let T denote the set of nonnegative integers n satisfying (3). Note that (1) implies:

(4) $$x_{2n+1} = 1 + \sqrt{1 + x_{2n}},\ x_{2n+2} = 1 - \sqrt{1 + x_{2n+1}},\ n = 0, 1, 2, \ldots .$$

Thus, since $-1 < x_0 < 0$, we have: $1 < x_1 < 2$. Hence, $0 \in T$. Assuming $k \in T$, by (4) we have:

$$1 - \sqrt{3} < x_{2k+2} < 1 - \sqrt{2} \Longrightarrow -1 < x_{2k+2} < 0 \Longrightarrow 1 < x_{2k+3} < 2;$$

i.e., $k \in T \Longrightarrow (k + 1) \in T$. By induction, (3) is proved. Now define

(5) $$a_n = x_{2n+1} - \alpha,\ b_n = x_{2n} - \beta,\ n = 0, 1, 2, \ldots .$$

Then, using (4), we obtain:

$$|a_n| = |\beta + \sqrt{1 + x_{2n}}| = |\beta + \sqrt{\beta^2 + b_n} = \alpha^{-1}|1 - (1 + \alpha^2 b_n)^{\frac{1}{2}}|$$
$$= \frac{\alpha^{-1}|-\alpha^2 b_n|}{|1 + (1 + \alpha^2 b_n)^{\frac{1}{2}}|} = \frac{|b_n|\alpha}{|1 + (1 + \alpha^2 b_n)^{\frac{1}{2}}|}.$$

However, using (3) and (5),

$$-\beta^2 = -1 - \beta < b_n < -\beta.$$

Hence,

$$0 < 1 + \alpha^2 b_n < 1 + \alpha = \alpha^2 \quad \text{and} \quad 1 < |1 + (1 + \alpha^2 b_n)^{\frac{1}{2}}| < \alpha.$$

Therefore,

(6) $$|a_n| < \alpha|b_n|.$$

Also,

$$|b_n| = |\alpha - \sqrt{1 + x_{2n-1}}| = |\alpha - \sqrt{\alpha^2 + a_{n-1}}| = \alpha|1 - (1 + \beta^2 a_{n-1})^{\frac{1}{2}}|$$
$$= \frac{\alpha|-\beta^2 a_{n-1}|}{|1 + (1 + \beta^2 a_{n-1})^{\frac{1}{2}}|} = \frac{\alpha^{-1}|a_{n-1}|}{|1 + (1 + \beta^2 a_{n-1})^{\frac{1}{2}}|}.$$

Again using (3) and (5), we have

$$\beta < a_{n-1} < \beta^2;$$

hence

$$1 + \beta^3 < 1 + \beta^2 a_{n-1} < 1 + \beta^4,$$

or

$$2\beta^2 < 1 + \beta^2 a_{n-1} < 3\beta^2 \Longrightarrow 1 + \sqrt{2}\alpha^{-1} < |1 + (1 + \beta^2 a_{n-1})^{\frac{1}{2}}|.$$

Thus,

$$|b_n| < \frac{|a_{n-1}|}{\alpha + \sqrt{2}};$$

since $\alpha + \sqrt{2} > 3$, thus

(7) $$|b_n| < \frac{|a_{n-1}|}{3}.$$

It follows from (6) and (7) that

$$|a_n| < \frac{\alpha}{3}|a_{n-1}| < .6|a_{n-1}| \quad \text{and} \quad |b_n| < \frac{\alpha}{3}|b_{n-1}| < .6|b_{n-1}| \quad (n = 1, 2, 3, \ldots).$$

Note that

$$-1 - \beta < x_0 - \beta < -\beta \Longrightarrow -1 < -\beta^2 < b_0 < \alpha^{-1} < 1 \Longrightarrow |b_0| < 1;$$

also,

$$1 - \alpha < x_1 - \alpha < 2 - \alpha \Longrightarrow \beta < a_0 < \beta^2 \Longrightarrow |a_0| < 1.$$

Therefore, by an easy induction, $|a_n| < (.6)^n$ and $|b_n| < (.6)^n$, which implies

$$|a_n| \to 0 \quad \text{and} \quad |b_n| \to 0$$

and hence

$$a_n \to 0 \quad \text{and} \quad b_n \to 0.$$

This, in turn, implies (2).

Note: The condition $x_0 \in (-1, 0)$ is sufficient but not necessary for the stated result to follow. It is only necessary that $x_0 \in [-1, 3]$.

Factor

H-315 *Proposed by D. P. Laurie, National Research Institute for Mathematical Sciences, Pretoria, South Africa (Vol. 18, No. 2, April 1980)*

Let the polynomial P be given by

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

and let $z_1, z_2, \ldots, z_n$ be distinct complex numbers. The following iteration scheme for factorizing P has been suggested by Kerner [1]:

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)}; \; i = 1, 2, \ldots, n.$$

Prove that if $\sum_{i=1}^{n} z_i = -a_{n-1}$, then also $\sum_{i=1}^{n} \hat{z}_i = -a_{n-1}$.

Reference: [1] I. Kerner. "Ein Gesamtschrittverfahren zur Berechnung der Null-stellen von Polynomen." *Numer. Math.* 8 (1966):290-94.

Solution by the proposer

Let

$$R(z) = P(z) - \prod_{i=1}^{n} (z - z_i),$$

since $\sum_{i=1}^{n} z_i = -a_{n-1}$, $R(z)$ is a polynomial of degree $n - 2$. We have

$$\sum_{i=1}^{n} \hat{z}_i = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} \frac{P(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)} = \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} \frac{R(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)}.$$

The second sum on the right is equal to the nth divided difference of R at the points $z_1, z_2, \ldots, z_n$ (see Davis [1], p. 40), and thus zero, since R is only of degree $n - 2$.

Reference: [1] P. Davis. *Interpolation and Approximation*. Blaisdell, 1963.

Also solved by Paul S. Bruckman.

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