Results similar to those obtained for $p_n^\lambda(x)$ may be obtained for $q_n^\lambda(x)$. At this stage, it is not certain just how useful a study of $q_n^\lambda(x)$ and $r_n^\lambda(x)$ might be.

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ENUMERATION OF PERMUTATIONS BY SEQUENCES—II

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1. André [1] discussed the enumeration of permutations by number of sequences; his results are reproduced in Netto's book [5, pp. 105-12]. Let $P(n, s)$ denote the number of permutations of $Z_n = \{1, 2, \ldots, n\}$ with s ascending or descending sequences. It is convenient to put

$$P(0, s) = P(1, s) = \delta_{0,s}. \tag{1.1}$$

André proved that $P(n, s)$ satisfies

$$P(n+1, s) = sP(n, s) + 2P(n, s-1) + (n-s+1)P(n, s-2), \quad (n \geq 1). \tag{1.2}$$

The following generating function for $P(n, s)$ was obtained in [2]:

$$\sum_{s=0}^{\infty}(1-x^2)^{-n/2}\frac{z^n}{n!}\sum_{s=0}^{n}P(n+1, s)x^{n-s} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2. \tag{1.3}$$

However, an explicit formula for $P(n, s)$ was not found.

In the present note, we shall show how an explicit formula for $P(n, s)$ can be obtained. We show first that the polynomial

$$p_n(x) = \sum_{s=0}^{n} P(n+1, x)(-x)^{n-s} \tag{1.4}$$

satisfies

$$p_{2n}(x) = \frac{1}{2^{n-1}}(1-x)^{n-1}\left\{2\sum_{k=1}^{n}(-1)^{n+k}A_{2n+1,k}T_{n-k+1}(x) - A_{2n+1,n+1}\right\} \tag{1.5}$$

and

$$p_{2n-1}(x) = \frac{1}{2^{n-2}}(1 - x)^{n-2}\sum_{k=0}^{n-1}(-1)^{k-1}(A_{2n,k} + A_{2n,k+1})T_{n-k}(x), \tag{1.6}$$

where the $A_{n,k}$ are the Eulerian numbers [3], [7, p. 240] defined by

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}A_{n,k}x^k = \frac{1 - x}{1 - xe^{z(1-x)}} \tag{1.7}$$

and $T_n(x)$ is the Chebychev polynomial of the first kind defined by [6, p. 301]

$$T_n(x) = \cos n\phi, \; x = \cos\phi. \tag{1.8}$$

Making use of (1.5) and (1.6), explicit formulas for $P(n, s)$ are obtained. For the final results, see (3.7), (3.8), and (4.2), (4.3).

2. In (1.3) take $x = -\cos\phi$, so that

$$\sum_{n=0}^{\infty}(\sin\phi)^{-n}\frac{z^n}{n!}\sum_{s=0}^{n}P(n + 1, s)(-\cos\phi)^{n-s} = \frac{1 + \cos\phi}{1 - \cos\phi}\left(\frac{\sin\phi + \sin z}{\cos\phi + \cos z}\right)^2. \tag{2.1}$$

We have

$$\left(\frac{\sin\phi + \sin z}{\cos\phi + \cos z}\right)^2 = \tan^2\frac{1}{2}(z + \phi) = \frac{1 - \cos(z + \phi)}{1 + \cos(z + \phi)}.$$

Hence, if we put

$$\left(\frac{\sin\phi + \sin z}{\cos\phi + \cos z}\right)^2 = \sum_{n=0}^{\infty}f_n(\cos\phi)\frac{z^n}{n!}, \tag{2.2}$$

it is clear that

$$f_n(\cos\phi) = \frac{d^n}{d\phi^n}\frac{1 - \cos\phi}{1 + \cos\phi}. \tag{2.3}$$

To evaluate this derivative, write

$$-\frac{1 - \cos\phi}{1 + \cos\phi} = \left(\frac{e^{\phi i} - 1}{e^{\phi i} + 1}\right)^2 = 1 - \frac{4}{e^{\phi i} + 1} + \frac{4}{(e^{\phi i} + 1)^2}.$$

Then

$$-\frac{1}{4}\frac{d}{d\phi}\frac{1 - \cos\phi}{1 + \cos\phi} = \frac{ie^{\phi i}}{(e^{\phi i} + 1)^2} - \frac{2ie^{\phi i}}{(e^{\phi i} + 1)^3} = \frac{i}{e^{\phi i} + 1} - \frac{3i}{(e^{\phi i} + 1)^2} + \frac{2i}{(e^{\phi i} + 1)^3}$$

and

$$-\frac{1}{4}\frac{d^2}{d\phi^2}\frac{1 - \cos\phi}{1 + \cos\phi} = \frac{i^2}{e^{\phi i} + 1} - \frac{7i^2}{(e^{\phi i} + 1)^2} + \frac{12i^2}{(e^{\phi i} + 1)^3} - \frac{6i^2}{(e^{\phi i} + 1)^4}.$$

The general formula is

$$(-1)^n\frac{1}{4}\frac{d^{n-2}}{d\phi^{n-2}}\frac{1 - \cos\phi}{1 + \cos\phi} = i^{n-2}\sum_{k=1}^{n}(-1)^{k-1}\frac{(k - 1)!S(n, k)}{(e^{\phi i} + 1)^k}, \quad (n > 2), \tag{2.4}$$

where $S(n, k)$ is the Stirling number of the second kind [7, Ch. 2]:

$$\sum_{n=k}^{\infty}S(n, k)\frac{z^n}{n!} = \frac{1}{k!}(e^z - 1)^k.$$

The proof of (2.4) by induction is simple. The derivative of the right-hand side is equal to

$$i^{n-1}\sum_{k=1}^{n}(-1)^k\,\frac{k!\;S(n,\,k)e^{\phi i}}{(e^{\phi i}+1)^{k+1}} = i^{n-1}\sum_{k=1}^{n}(-1)^k\left\{\frac{k!\;S(n,\,k)}{(e^{\phi i}+1)^k} - \frac{k!\;S(n,\,k)}{(e^{\phi i}+1)^{k+1}}\right\}$$

$$= i^{n-1}\sum_{k=1}^{n+1}(-1)^k\frac{(k-1)!}{(e^{\phi i}+1)^k}\{(kS(n,\,k)+S(n,\,k-1)\}.$$

Since $kS(n,\,k) + S(n,\,k-1) = S(n+1,\,k)$, this evidently completes the induction.

We may rewrite (2.4) in the following form:

$$\frac{1}{4}\,\frac{d^{n-2}}{d\phi^{n-2}}\,\frac{1-\cos\phi}{1+\cos\phi} = \frac{(-1)^n i^{n-2}}{(e^{\phi i}+1)^n}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n,\,k)(e^{\phi i}+1)^{n-k},\quad (n>2). \tag{2.5}$$

In the next place, we require the identity

$$\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n,\,k)(x+1)^{n-k} = \sum_{k=0}^{n-1}(-1)^{n-k-1}A_{n-1,k}\,x^k,\quad (n\geq 1), \tag{2.6}$$

where $A_{n-1,k}$ is the Eulerian number defined by (1.7).

To prove (2.6), take

$$\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n,\,k)(x+1)^{n-k}$$

$$= \sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!(x+1)^{-k}\sum_{n=k}^{\infty}S(n,\,k)\frac{z^n(x+1)^n}{n!}$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}(x+1)^{-k}\left(e^{z(x+1)}-1\right)^k$$

$$= \log\left(1+\frac{e^{z(x+1)}-1}{x+1}\right) = \log\frac{x+e^{z(x+1)}}{x+1}.$$

Differentiating with respect to z, we get

$$\sum_{n=\infty}^{\infty}\frac{z^{n-1}}{(n-1)!}\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n,\,k)(x+1)^{n-k} = \frac{(x+1)e^{z(x+1)}}{x+e^{z(x+1)}} = \frac{1+x}{1+xe^{-z(1+x)}}.$$

On the other hand, by (1.7),

$$\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!}\sum_{k=0}^{n}(-1)^kA_{n,k}\,x^k = \frac{1+x}{1+xe^{-z(1+x)}}.$$

Hence,

$$\sum_{k=1}^{n}(-1)^{k-1}(k-1)!S(n,\,k)(x+1)^{n-k} = \sum_{k=0}^{n-1}(-1)^{n-k-1}A_{n-1,k}\,x^k.$$

3. By (2.5) and (2.6) we have, on replacing n by $n+2$,

$$\frac{1}{4}\,\frac{d}{d\phi}\,\frac{1-\cos\phi}{1+\cos\phi} = \frac{(-1)^n}{(e^{\phi i}+1)^{n-2}}\sum_{k=1}^{n+1}(-1)^{n-k+1}A_{n+1,\,k}e^{k\phi i},$$

since $A_{n+1,0} = 0$. Moreover, since [3]

$$A_{n+1,k} = A_{n+1,n-k+2} \qquad (1 \le k \le n+1), \tag{3.1}$$

we have

$$\begin{aligned}\frac{1}{4}\frac{d^n}{d\phi^n}\frac{1-\cos\phi}{1+\cos\phi} &= \frac{1}{2}(-1)^n \sum_{k=1}^{n+1}(-1)^{k+1}A_{n+1,k}\frac{e^{(n-k+2)\phi i} + (-1)^n e^{k\phi i}}{(e^{\phi i}+1)^{n+2}} \\ &= \frac{1}{2}(-i)^n \sum_{k=1}^{n+1}(-1)^{k+1}A_{n+1,k}\frac{e^{\frac{1}{2}(n-2k+2)\phi i} + (-1)^n e^{-\frac{1}{2}(n-2k+2)\phi i}}{(e^{\frac{1}{2}\phi i} + e^{-\frac{1}{2}\phi i})^{n+2}}.\end{aligned}$$

Therefore, in view of (2.3), we get

$$f_n(\cos\phi) = 2(-i)^n \sum_{k=1}^{n+1}(-1)^{k+1}A_{n+1,k}\frac{e^{\frac{1}{2}(n-2k+2)\phi i} + (-1)^n e^{-\frac{1}{2}(n-2k+2)\phi i}}{\left(2\cos\frac{1}{2}\phi\right)^{n+2}}. \tag{3.2}$$

It is convenient to consider n even and n odd separately, so that

$$f_{2n}(\cos\phi) = \frac{1}{2^{2n}}\sum_{k=1}^{2n+1}(-1)^{n+k+1}A_{2n+1,k}\frac{\cos(n-k+1)\phi}{\left(\cos\frac{1}{2}\phi\right)^{2n+2}} \tag{3.3}$$

and

$$f_{2n-1}(\cos\phi) = \frac{1}{2^{2n-1}}\sum_{k=1}^{2n}(-1)^{n+k}A_{2n,k}\frac{\sin\frac{1}{2}(2n-2k+1)\phi}{(\cos\phi)^{2\quad 1}}. \tag{3.4}$$

By (1.3), (1.4), and (2.2),

$$\begin{aligned}p_n(\cos\phi) &= \frac{1+\cos\phi}{1-\cos\phi}\sin^n\phi\, f_n(\cos\phi) \\ &= 2^n \cos^{n+2}\tfrac{1}{2}\phi\ \sin^{n-2}\tfrac{1}{2}\phi\ f_n(\cos\phi).\end{aligned} \tag{3.5}$$

In particular

$$p_{2n}(\cos\phi) = 2^{2n}\cos^{2n+2}\tfrac{1}{2}\phi\ \sin^{2n-2}\tfrac{1}{2}\phi\ f_{2n}(\cos\phi),$$

so that, by (3.3),

$$p_{2n}(\cos\phi) = \frac{1}{2^{n-1}}(1-\cos\phi)^{n-1}\cdot\sum_{k=1}^{2n+1}(-1)^{n+k+1}A_{2n+1,k}\cos(n-k+1)\phi. \tag{3.6}$$

Using (3.1) and (.18), (3.6) gives

$$p_{2n}(x) = \frac{1}{2^{n-1}}(1-x)^{n-1}\left\{2\sum_{k=1}^{n}(-1)^{n+k+1}A_{2n+1,k}T_{n-k+1}(x) + A_{2n+1,n+1}\right\}. \tag{3.7}$$

This proves (1.5).

Next, replacing n by $2n-1$ in (3.5), we get

$$\begin{aligned}p_{2n-1}(\cos\phi) &= 2^{2n-1}\cos^{2n+1}\tfrac{1}{2}\phi\ \sin^{2n-3}\tfrac{1}{2}\phi\ f_{2n-1}(\cos\phi) \\ &= \sin^{2n-3}\tfrac{1}{2}\phi\sum_{k=1}^{2n}(-1)^{n+k}A_{2n,k}\sin\tfrac{1}{2}(2n-2k+1)\phi\end{aligned}$$

(continued)

$$= \frac{1}{2}\sin^{2n-4}\frac{1}{2}\phi\sum_{k=1}^{2n}(-1)^{n+k}A_{2n,k}\{\cos(n-k)\phi - \cos(n-k+1)\phi\}$$

$$= \frac{1}{2^{n-1}}(1-\cos\phi)^{n-2}\sum_{k=1}^{2n}(-1)^{n+k}(A_{2n,k}+A_{2n,k+1})\cos(n-k)\phi$$

$$= \frac{1}{2^{n-2}}(1-\cos\phi)^{n-2}\left\{\sum_{k=0}^{n-1}(-1)^{n+k}(A_{2n,k}+A_{2n,k+1})\cos(n-k)\phi + A_{2n,n}\right\}.$$

Finally, therefore, by (1.8),

$$p_{2n-1}(x) = \frac{1}{2^{n-2}}(1-x)^{n-2}\left\{\sum_{k=0}^{n-1}(-1)^{n+k}(A_{2n,k}+A_{2n,k+1})T_{n-k}(x) + A_{2n,n}\right\}. \tag{3.8}$$

4. We recall that

$$T_n(x) = \frac{1}{2}\sum_{0<2j\le n}(-1)^j\left\{\binom{n-j}{j}+\binom{n-j-1}{j-1}\right\}(2x)^{n-2j} \tag{4.1}$$

$$= 2^{n-1}x^n + \frac{1}{2}\sum_{0<2j\le n}(-1)^j\frac{n}{j}\binom{n-j-1}{j-1}(2x)^{n-2j}, \qquad (n \ge 1).$$

Thus (3.7) becomes

$$p_{2n}(x) = \frac{1}{2^{n-1}}\sum_{t=0}^{n-1}(-1)^t\binom{n-1}{t}x^t\cdot\left\{\sum_{k=1}^{n}(-1)^{n+k-1}A_{2n+1,k}\sum_{2j\le n-k+1}(-1)^j\right.$$

$$\left.\cdot\left[\binom{n-k-j+1}{j}+\binom{n-k-j}{j-1}\right](2x)^{n-k-2j+1} + A_{2n+1,n+1}\right\}$$

$$= \frac{1}{2^{n-1}}\sum_{s=0}(-x)^{2n-s}\sum_{k=1}^{n}A_{2n+1,k}\sum_{s=n+k+2j-t-1}(-1)^j\left\{\binom{n-k-j+1}{j}\right.$$

$$\left.+\binom{n-k-j}{j-1}\right\}\binom{n-1}{t}2^{n-k-2j+1} + \frac{1}{2^{n-1}}A_{2n+1,n+1}\sum_{s=n+1}^{2n}\binom{n-1}{2n-s}(-x)^{2n-s}.$$

Comparison with (1.4) gives

$$P(2n+1,s) = \frac{1}{2^{n-1}}\sum_{k=1}^{n}A_{2n+1,k}\sum_{s=n+k+2j-t-1}(-1)^j\left\{\binom{n-k-j+1}{j}\right. \tag{4.2}$$

$$\left.+\binom{n-k-j}{j-1}\right\}\binom{n-1}{t}2^{n-k-2j+1} + \frac{1}{2^{n-1}}\binom{n-1}{2n-s}A_{2n+1,n+1}.$$

Similarly, it follows from (3.8) that

$$P(2n,s) = \frac{1}{2^{n-2}}\sum_{k=0}^{n-1}(A_{2n,k}+A_{2n,k+1})\sum_{s=n+k+2j-t-1}(-1)^j \tag{4.3}$$

$$\cdot\left\{\binom{n-k-j}{j}+\binom{n-k-j-1}{j-1}\right\}\binom{n-2}{t}2^{n-k-2j}$$

$$+\frac{1}{2^{n-2}}\binom{n-2}{2n-s-1}A_{2n,n}.$$

5. For numerical checks of the above results, it is probably easier to use (3.7) and (3.8) rather than the explicit formulas (4.2) and (4.3).

It is convenient to recall the following tables for $P(n, s)$ and $A_{n,k}$, respectively:

TABLE 1

n \ s	0	1	2	3	4	5	6
1	1						
2		2					
3		2	4				
4		2	12	10			
5		2	28	58	32		
6		2	60	236	300	122	
7		2	124	836	1852	1682	544

TABLE 2

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

We first take (3.7) with $n = 2$. Then

$$\begin{aligned} p_4(x) &= \frac{1}{2}(1 - x)\{2A_{5,1}T_2(x) - 2A_{5,2}T_1(x) + A_{5,3}\} \\ &= \frac{1}{2}(1 - x)\{2(2x^2 - 1) + 52x + 66\} \\ &= 2x^3 - 28x^2 + 58x - 32. \end{aligned}$$

Taking $n = 3$ in (3.7), we get

$$\begin{aligned} p_6(x) &= \frac{1}{4}(1 - x)^2\{-2A_{7,1}T_3(x) + 2A_{7,2}T_2(x) - 2A_{7,3}T_1(x) + A_{7,4}\} \\ &= \frac{1}{4}(1 - x)^2\{-2(4x^3 - 3x) + 2 \cdot 120(2x^2 - 1) - 2 \cdot 1191x + 2416\} \\ &= (1 - x)^2(544 - 1188x + 120x^2 - 2x^3) \\ &= 544 - 1682x + 1852x^2 - 836x^3 + 124x^4 - 2x^5. \end{aligned}$$

Next, taking $n = 2$ in (3.8), we get

$$\begin{aligned} p_3(x) &= \sum_{k=0}^{1}(-1)^k(A_{4,k} + A_{4,k+1})T_{2-k}(x) + A_{4,2} \\ &= A_{4,1}T_2(x) - (A_{4,1} + A_{4,2})T_1(x) + A_{4,2} \\ &= (2x^2 - 1) - 12x + 11 \\ &= 2x^2 - 12x + 10. \end{aligned}$$

Similarly, taking $n = 3$ in (3.8), we get

$$\begin{aligned} p_5(x) &= \frac{1}{2}(1 - x)\left\{\sum_{k=0}^{2}(-1)^{3+k}(A_{6,k} + A_{6,k+1})T_{3-k}(x) + A_{6,3}\right\} \\ &= \frac{1}{2}(1 - x)\{-A_{6,1}T_3(x) + (A_{6,1} + A_{6,2})T_2(x) - (A_{6,2} + A_{6,3})T_1(x) + A_{6,3}\} \\ &= \frac{1}{2}(1 - x)\{-(4x^3 - 3x) + 58(2x^2 - 1) - 359x + 302\} \\ &= 2x^4 - 60x^3 + 236x^2 - 300x + 122. \end{aligned}$$

Another partial check is furnished by taking $x = -1$ in (3.7) and (3.8). Since $T_n(-1) = \cos n\pi = (-1)^n$, it is easily verified that (3.7) and (3.8) reduce to

$$p_{2n}(-1) = 2\sum_{k=1}^{n}(A_{2n+1,k} + A_{2n+1,n+1}) = \sum_{k=1}^{2n+1}A_{2n+1,k} = (2n + 1)!$$

and

$$p_{2n-1}(-1) = \sum_{k=0}^{n-1}(A_{2n,k} + A_{2n,k+1}) + A_{2n,n} = \sum_{k=1}^{2n}A_{2n,k} = (2n)!,$$

respectively.

On the other hand, for $x = 1$, it is evident from (3.7) and (3.8) that

$$p_n(1) = 0 \qquad (n \geq 4). \tag{5.1}$$

Moreover, since $T_n(1) = 1$, it follows from (3.7) and (3.1) that

$$\begin{aligned} p_{2n}^{(n+1)}(1) &= (-1)^{n-1}\frac{(n - 1)!}{2^{n-1}}\left\{2\sum_{k=1}^{n}(-1)^{n+k+1}A_{2n+1,k} + A_{2n+1,n+1}\right\} \\ &= \frac{(n - 1)!}{2^{n-1}}\sum_{k=1}^{2n+1}(-1)\ A_{2n+1,k}. \end{aligned}$$

By (1.7), we have

$$1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{k=1}^{n} (-1)^k A_{n,k} = \frac{2}{e^{2z} + 1} = \sum_{n=0}^{\infty} C_n \frac{z^n}{n!}$$

in the notation of Nörlund [5, p. 27]. Hence

$$p_{2n}^{(n-1)}(1) = \frac{(n-1)!}{2^{n-1}} C_{2n+1}. \tag{5.2}$$

For example

$$p_6''(1) = 3704 - 5016 + 1488 - 40 = 136;$$

since $C_7 = 272$, this is in agreement with (5.2).

As for $p_{2n-1}(x)$, it follows from (3.8) that

$$\begin{aligned} p_{2n-1}^{(n-2)}(1) &= (-1)^{n-2} \frac{(n-2)!}{2^{n-2}} \left\{ \sum_{k=0}^{n-1} (-1)^{n+k} (A_{2n,k} + A_{2n,k+1}) + A_{2n,n} \right\} \\ &= \frac{(n-2)!}{2^{n-2}} \left\{ \sum_{k=1}^{n} (-1)^{k-1} A_{2n,k} + \sum_{k=1}^{n-1} (-1)^k A_{2n-k+1} + (-1)^n A_{2n,n} \right\} \\ &= \frac{(n-2)!}{2^{n-2}} \sum_{k=1}^{2n} (-1)^{k-1} A_{2n,k}, \end{aligned}$$

so that

$$p_{2n-1}^{(n-2)}(1) = 0 \qquad (n \geq 2). \tag{5.3}$$

Next, we have

$$p_{2n-1}^{(n-1)}(1) = (-1)^{n-2} \frac{(n-1)!}{2^{n-2}} \left\{ \sum_{k=0}^{n-1} (-1)^{n+k} (A_{2n,k} + A_{2n,k+1}) \right\} T'_{n-k}(1).$$

By (1.8),

$$T_n'(x) = \frac{n \sin n\phi}{\sin \phi} \qquad (x = \cos \phi),$$

which gives $T_n'(1) = n^2$. Thus

$$p_{2n-1}^{(n-1)}(1) = \frac{(n-1)!}{2^{n-2}} \sum_{k=0}^{n-1} (-1)^k (n-k)^2 (A_{2n,k} + A_{2n,k+1}).$$

After some manipulation, we get

$$\begin{aligned} p_{2n-1}^{(n-1)}(1) &= \frac{(n-1)!}{2^{n-2}} \sum_{k=1}^{n} (-1)^{k-1} (2n - 2k + 1) A_{2n,k} \\ &= \frac{(n-1)!}{2^{n-2}} \sum_{k=1}^{2n} (-1)^k k A_{2n,k}. \end{aligned} \tag{5.4}$$

Making use of (1.7), it can be proved that

$$(1 - x)A_n'(x) = A_{n+1}(x) - (n+1)xA_n(x), \tag{5.5}$$

where

$$A_n(x) = \sum_{k=1}^{n} A_{nk} x^k \qquad (n \geq 1).$$

Hence

$$2A_{2n}'(-1) = A_{2n+1}(-1) + (2n+1)A_{2n}(-1) = C_{2n+1},$$

where C_{2n+1} has the same meaning as above. Thus (5.4) reduces to

$$p_{2n-1}^{(n-1)}(1) = \frac{(n-1)!}{2^{n-1}} C_{2n+1}. \tag{5.6}$$

For example,

$$p_5''(1) = 24 - 360 + 472 = 136,$$

in agreement with (5.6).

(Please turn to page 465.)

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HOW TO FIND THE "GOLDEN NUMBER" WITHOUT REALLY TRYING

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". . .I wish . . . to point out that the use of the golden section . . . has apparently burst out into a sudden and devastating disease which has shown no signs of stopping . . ." [2, p. 521]

Most of the papers involving claims concerning the "golden number" deal with distinct items such as paintings, basing their assertions on measurements of these individual objects. As an example, we may cite the article by Hedian [13]. However measurements, no matter how accurate, cannot be used to reconstruct the original system of proportions used to design an object, for many systems may give rise to approximately the same set of numbers; see [6, 7] for an example of this. The only valid way of determining the system of proportions used by an artist is by means of documentation. A detailed investigation of three cases [8, 9, 10, 11] for which it had been claimed in the literature that the artist in question had used the "golden number" showed that these assertions were without any foundation whatsoever.

There is, however, another class of papers that seeks to convince the reader via statistical data applied to a whole class of related objects. The earliest examples of these are Zeising's morphological works, e.g., [17]. More recently we have Duckworth's book [5] on Vergil's *Aeneid* and a series of papers by Benjafield and his coauthors involving such things as interpersonal relationships (see e.g. [1], which gives a partial listing of some of these papers).

Mathematically we may approach the question in the following way. Suppose we have a certain length which is split into two parts, the larger being M and the smaller m. If the length is divided according to the golden section, then it does not matter which of the quantities, m/M or $M/(M + m)$, we use, for they are equal. But now suppose we have a collection of lengths and we are trying to determine statistically if the data are consistent with a partition according to the golden