

ON SOME EXTENSIONS OF THE MEIXNER-WEISNER GENERATING FUNCTIONS

M. E. COHEN

California State University, Fresno, CA 93710

H. S. SUN

Academia Sinica, Taipei, China

I. INTRODUCTION

With the aid of group theory, Weisner [10] derived the Bilinear generating function for the ultraspherical polynomial:

$$(1.1) \quad \sum_{n=0}^{\infty} \frac{n! t^n}{(2\alpha)_n} C_n^\alpha(\cos x) C_n^\alpha(\cos y) \\ = \{1 - 2t \cos(x + y) + t^2\}^{-\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \alpha; \\ 2\alpha; \end{matrix} \frac{4t \sin x \sin y}{1 - 2t \cos(x + y) + t^2} \right]$$

See [5] for definition and properties. (1.1) had also been proved by Meixner [6], Ossicini [7], and Watson [8], and was recently investigated by Carlitz [2], [3]. (1.1) is seen to be a special case of Theorem 1 in this paper, as are the formulas (1.2), (1.4), and (1.5), which appear to be new. Note that the expressions given below are generating functions for the ultraspherical polynomial of type $C_{n+\ell}^\lambda(x)$. See Cohen [4] for the single Jacobi polynomial.

$$(1.2) \quad \sum_{n=0}^{\infty} \frac{t^n (n + \ell)!}{(2u + 2\ell + 1)_n} C_n^{u+\ell+\frac{1}{2}}(x) C_n^u(y) \\ = \frac{2^{\ell+1} \Gamma(u + \ell + 1) \ell!}{\Gamma(u) \Gamma(2u + \ell) [t^2(x^2 - 1)]^{u+\frac{1}{2}\ell}} C_\ell^u \left[\left\{ \frac{2(y - xt)^2}{2x^2 t^2 - 2xyt - t^2 + 1 + \rho} \right\}^{\frac{1}{2}} \right] D_\ell^u \\ \cdot \left[\left\{ \frac{2x^2 t^2 - 2xyt - t^2 + 1 + \rho}{2t^2(x^2 - 1)} \right\}^{\frac{1}{2}} \right]$$

where $\rho = [(1 - 2xyt + t^2)^2 - 4t^2(1 - x^2)(1 - y^2)]^{\frac{1}{2}}$, $|t| < 1$, $|xt/y| < 1$, ℓ is a nonnegative integer, and D_ℓ^u is the Gegenbauer function defined by Watson [9, p. 129] as

$$(1.3) \quad D_\ell^u(z) = \frac{\Gamma(u) \Gamma(2u + \ell) z^{-2u-\ell}}{2^{\ell+1} \Gamma(u + \ell + 1)} {}_2F_1 \left[\begin{matrix} u + \frac{1}{2}\ell, u + \frac{1}{2}\ell + \frac{1}{2}; \\ u + \ell + 1; \end{matrix} \frac{1}{z^2} \right].$$

A special case of (1.2) is deduced for $x = 0$, $y = \cos \phi$, and t suitably modified:

$$(1.4) \quad \sum_{n=0}^{\infty} \frac{t^n (2n + \ell)!}{2^{2n} n! (v + \ell + 1)_n} C_{2n+\ell}^v(\cos \phi) \\ = \frac{\ell! \Gamma(v + \ell + 1) 2^{\ell+1}}{\Gamma(v) \Gamma(2v + \ell) t^{v+\frac{1}{2}\ell}} C_\ell^v \left[\left\{ \frac{2 \cos^2 \phi}{1 + t + \sigma} \right\}^{\frac{1}{2}} \right] D_\ell^v \left[\left\{ \frac{1 + t + \sigma}{2t} \right\}^{\frac{1}{2}} \right]$$

where $\sigma = (1 - 2t \cos(2\phi) + t^2)^{\frac{1}{2}}$, and $|t| < 1$.

$$(1.5) \quad \frac{\sum_{n=0}^{\infty} t^n C_n^{u+\frac{1}{2}}(\cos \theta) C_{n+1}^u(\cos \phi)}{C_{n+1}^u(0)} = (\cos \phi - t \cos \theta)^{-1-2u} (1 - \xi \eta)^{-1} (1 - \xi)^{u+\frac{1}{2}} (1 - \eta)^{u+1}$$

where $|t \cos \theta / \cos \phi| < 1$, $|t| < 1$,

$$\xi = \{1 - 2t \cos \theta \cos \phi + t^2 - [(1 - 2t \cos(\theta - \phi) + t^2)(1 - 2t \cos(\theta + \phi) + t^2)]^{\frac{1}{2}}\} / 2 \sin^2 \phi,$$

$$\eta = \{1 - 2t \cos \theta \cos \phi + t^2 - [(1 - 2t \cos(\theta - \phi) + t^2)(1 - 2t \cos(\theta + \phi) + t^2)]^{\frac{1}{2}}\} / 2t^2 \sin^2 \theta.$$

Theorem 2 yields the new finite expansions

$$(1.6) \quad \sum_{n=0}^{\ell} \frac{(-1)^n t^n}{(2v)_n (1 - 2\ell - 2v)_{\ell-n}} C_n^v(x) C_{\ell-n}^{\frac{1}{2}-v-\ell}(y) = \frac{\ell! (y^2 - 1)^{\frac{1}{2}\ell}}{2^\ell (v)_\ell (2v)_\ell} C_\ell^v \left[\left\{ \frac{2(y-xt)^2}{2y^2 - 2xyt + t^2 - 1 + \rho} \right\}^{\frac{1}{2}} \right] C_\ell^v \left[\left\{ \frac{2(y-xt)^2}{2y^2 - 2xyt + t^2 - 1 - \rho} \right\}^{\frac{1}{2}} \right]$$

where ρ is defined in equation (1.2).

Equation (1.6) may also be expressed as

$$(1.7) \quad \sum_{n=0}^{\ell} \frac{t^n 2^n}{(2v)_n (v)_{\ell-n}} C_n^v(x) C_{\ell-n}^{v+n}(y') = \frac{\ell!}{(v)_\ell (2v)_\ell} C_\ell^v \left[\left\{ \frac{y'^2 + 1 + 2xy't' + t'^2 - \rho'}{2} \right\}^{\frac{1}{2}} \right] C_\ell^v \left[\left\{ \frac{y'^2 + 1 + 2xy't' + t'^2 + \rho'}{2} \right\}^{\frac{1}{2}} \right]$$

where $\rho' = \{(y'^2 - 1 + 2xy't' + t'^2)^2 + 4t'^2(1 - x^2)\}^{\frac{1}{2}}$.

A special case of (1.6) is the relation

$$(1.8) \quad \sum_{n=0}^{[\ell/2]} \frac{t^n}{2^{2n} (2v)_{\ell-2n} (1 - \ell - v)_n n!} C_{\ell-2n}^v(\cos \phi) = \frac{\ell! t^{\frac{1}{2}\ell}}{2^\ell (v)_\ell (2v)_\ell} C_\ell^v \left[\left\{ \frac{2 \cos^2 \phi}{1 + t + \sigma} \right\}^{\frac{1}{2}} \right] C_\ell^v \left[\left\{ \frac{2 \cos^2 \phi}{1 + t - \sigma} \right\}^{\frac{1}{2}} \right]$$

where σ is defined in equation (1.4).

Equation (1.8) is deduced from (1.6) by putting $x = 0$, and rearranging the parameters. Also, if $y = 1$ in (1.6), one obtains a known expression [5, p. 227, last formula].

SECTION II

Theorem 1: For u and v arbitrary complex numbers and ℓ a nonnegative integer,

$$\sum_{n=0}^{\infty} \frac{t^n (n + \ell)!}{(2v)_n} C_n^v(x) C_{n+\ell}^u(y)$$

$$(2.1) \quad = (2u)_\ell (y - xt)^{-2u - \ell} F_4 \left[\frac{1}{2}(2u + \ell), \frac{1}{2}(2u + \ell + 1); v + \frac{1}{2}, u + \frac{1}{2}; \frac{t^2(x^2 - 1)}{(y - xt)^2}, \frac{y^2 - 1}{(y - xt)^2} \right]$$

where $|xt/y| < 1$, $|t| < 1$, and F_4 denotes the fourth type of Appell's [1, p. 14] hypergeometric function of two variables defined by

$$F_4[a, b; c, d; x_1, y_1] = \sum_{k_1, k_2} \frac{(a)_{k_1 + k_2} (b)_{k_1 + k_2}}{k_1! k_2! (c)_{k_1} (d)_{k_2}} x_1^{k_1} x_2^{k_2}.$$

Proof: The left-hand side of (2.1) may be expressed as

$$(2.2) \quad \sum_{n=0}^{\infty} \frac{(2u)_{n+\ell} t^n x^n y^{n+\ell} (n+\ell)}{n!} {}_2F_1 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \\ v + \frac{1}{2}; \end{matrix} \frac{x^2 - 1}{x^2} \right] \\ \cdot {}_2F_1 \left[\begin{matrix} -\frac{1}{2}(n+\ell), -\frac{1}{2}(n+\ell) + \frac{1}{2}; \\ u + \frac{1}{2}; \end{matrix} \frac{y^2 - 1}{y^2} \right]$$

$$(2.3) \quad = \sum_{n=0}^{\infty} \frac{(2u)_{n+\ell} t^n x^n y^{n+\ell} (n+\ell)}{n!} {}_2F_1 \left[\begin{matrix} -\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \\ v + \frac{1}{2}; \end{matrix} \frac{x^2 - 1}{x^2} \right] y^{-2u - 2n - 2\ell} \\ \cdot {}_2F_1 \left[\begin{matrix} u + \frac{1}{2}(n+\ell), u + \frac{1}{2}(n+\ell) + \frac{1}{2}; \\ u + \frac{1}{2}; \end{matrix} \frac{y^2 - 1}{y^2} \right]$$

$$(2.4) \quad = \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \left[\frac{t^2(x^2 - 1)}{4y^2} \right]^k \left[\frac{y^2 - 1}{4y^2} \right]^p \frac{(2u)_\ell (2u + \ell)_{2p+2k} (n + \ell + 2k)}{y^{2u+\ell} p! k! \left(u + \frac{1}{2}\right)_p \left(v + \frac{1}{2}\right)_k} \\ \cdot \sum_{n=0}^{\infty} \frac{(xt/y)^n (2u + \ell + 2k + 2p)_n}{n!}$$

From (2.2) to (2.3) we have used the Kummer transformation. Going from (2.3) to (2.4) entails the use of the following:

$$(2.5) \quad 2^{2k} \left(-\frac{1}{2}n\right)_k \left(-\frac{1}{2}n + \frac{1}{2}\right)_k = \frac{n!}{(n - 2k)!}$$

$$(2.6) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} f(n, k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} f(n + 2k, k)$$

and

$$(2.7) \quad (2u)_{n+\ell+2k} (2u + n + \ell + 2k)_{2p} = (2u)_\ell (2u + \ell)_{2p+2k} (2u + \ell + 2p + 2k)_n.$$

Now

$$(2.8) \quad \sum_{n=0}^{\infty} \frac{(xt/y)^n (2u + \ell + 2k + 2p)_n}{n!} = [1 - xt/y]^{-2u - \ell - 2k - 2p}.$$

Hence, (2.4) reduces to

$$(2.9) \quad (y - xt)^{-2u - \ell} (2u)_\ell \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \left[\frac{t^2(x^2 - 1)}{(y - xt)^2} \right]^k \left[\frac{(y^2 - 1)}{(y - xt)^2} \right]^p$$

$$\cdot \frac{\left(\frac{1}{2}(2u + \ell)\right)_{p+k} \left(\frac{1}{2}(2u + \ell + 1)\right)_{p+k}}{k! p! \left(v + \frac{1}{2}\right)_k \left(u + \frac{1}{2}\right)_p}$$

By definition, (2.9) is the right-hand side of Theorem 1.

Theorem 2: For u and v arbitrary complex numbers and ℓ a nonnegative integer,

$$(2.10) \quad \sum_{n=0}^{\ell} \frac{(-1)^n t^n}{(2v)_n (2u)_{\ell-n}} C_n^v(x) C_{\ell-n}^v(y)$$

$$= \frac{(y - xt)^\ell}{\ell!} F_4 \left[-\frac{1}{2}\ell, -\frac{1}{2}\ell + \frac{1}{2}; v + \frac{1}{2}, u + \frac{1}{2}; \frac{t^2(x^2 - 1)}{(y + xt)^2}, \frac{y^2 - 1}{(y - xt)^2} \right].$$

Proof: The left-hand side of (2.10) is put in the form

$$(2.11) \quad \sum_{n=0}^{\ell} \frac{(-1)^n t^n x^n y^{\ell-n}}{n! (\ell - n)!} {}_2F_1 \left[-\frac{1}{2}n, -\frac{1}{2}n + \frac{1}{2}; \frac{x^2 - 1}{x^2} \right]$$

$$\cdot {}_2F_1 \left[-\frac{1}{2}(\ell - n), -\frac{1}{2}(\ell - n) + \frac{1}{2}; \frac{y^2 - 1}{y^2} \right].$$

Following a procedure analogous to that in the proof of Theorem 1, with appropriate changes, (2.11) is simplified to yield the right-hand side of (2.10).

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