ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-460 *Proposed by Larry Taylor, Rego Park, NY*

For all integers j, k, n, prove that

$$F_k F_{n+j} - F_j F_{n+k} = (-1)^j F_{k-j} F_n.$$

B-461 *Proposed by Larry Taylor, Rego Park, NY*

For all integers j, k, n, prove or disprove that

$$F_k L_{n+j} - F_j L_{n+k} = (-1)^j F_{k-j} L_n.$$

B-462 *Proposed by Herta T. Freitag, Roanoke, VA*

Let $L(n)$ denote L_n and $T_n = n(n+1)/2$. Prove or disprove:

$$L(n) = (-1)^{T_{n-1}} [L(T_{n-1})L(T_n) - L(n^2)].$$

B-463 *Proposed by Herta T. Freitag, Roanoke, VA*

Using the notations of B-462, prove or disprove:

$$L(n) \equiv (-1)^{T_{n-1}} L(n^2) \pmod 5.$$

B-464 *Proposed by Gregory Wulcyzn, Bucknell University, Lewisburg, PA*

Let n and w be integers with w odd. Prove or disprove:

$$F_{n+2w}F_{n+w} - 2L_w F_{n+w} F_{n-w} - F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

B-465 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For positive integers n and k, prove or disprove:

$$\frac{F_{2k} + F_{6k} + F_{10k} + \cdots + F_{(4n-2)k}}{L_{2k} + L_{6k} + L_{10k} + \cdots + L_{(4n-2)k}} = \frac{F_{2nk}}{L_{2nk}}$$

SOLUTIONS

Sequence Identified and Summed

B-436 *Proposed by Sahib Singh, Clarion State College, Clarion, PA*

Find an appropriate expression for the nth term of the following sequence and also find the sum of the first n terms:

$$4,\ 2,\ 10,\ 20,\ 58,\ 146,\ 388,\ 1001,\ \ldots\ .$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

The general term d_k of the above sequence is given by

$$d_k = F_k^2 + L_k^2,\ k = 0,\ 1,\ 2,\ \ldots\ .$$

Let S be the sum of its first n terms. Then

$$S = \sum_{k=0}^{n-1} F_k^2 + \sum_{k=0}^{n-1} L_k^2 = F_{n-1}F_n + L_{n-1}L_n + 2$$

[see (I_3) and (I_4) on p. 55 of *Fibonacci and Lucas Numbers* by V. E. Hoggatt, Jr.]

Also solved by Wray G. Brady, Lars Brodin, Paul S. Bruckman, Scott St. Michel and James F. Peters, A. G. Shannon, Charles B. Shields, M. Wachtel and E. Schmutz and H. Klauser, Gregory Wulczyn, and the proposer.

Hoggatt-Hansell Property

B-437 *Proposed by G. Iommi Amunategui, Universidad Católica de Valparaíso, Chile*

Let $[m,\ n] = mn(m + n)/2$ for positive integers m and n. Show that:

(a) $[m + 1,\ n][m,\ n + 2][m + 2,\ n + 1] = [m,\ n + 1][m + 2,\ n][m + 1,\ n + 2]$.

(b) $\displaystyle\sum_{k=1}^{m} [m + 1 - k,\ k] = m(m + 1)^2(m + 2)/12$.

(We note that part a is the Hoggatt-Hansell "Star of David" property for the $[m,\ n]$.)

Solution by Gregory Wulczyn, Bucknell University, Lewisburg, PA

(a)
$$\begin{aligned}[m + 1,\ n][m,\ n + 2][m + 2,\ n + 1] &= \tfrac{1}{8}m(m + 1)(m + 2)(m + n + 1)\\ &\qquad (m + n + 2)(m + n + 3)\\ &= [m,\ n + 1][m + 2,\ n][m + 1,\ n + 2].\end{aligned}$$

(b)
$$\begin{aligned}\sum_{k=1}^{m} [m + 1 - k,\ k] &= \frac{1}{2}(m + 1) \sum_{k=1}^{m} k(m + 1 - k)\\ &= \frac{m + 1}{2}\left[\frac{(m + 1)m(m + 1)}{2} - \frac{m(m + 1)(2m + 1)}{6}\right]\\ &= \frac{m(m + 1)^2}{12}[3m + 3 - 2m - 1] = \frac{m(m + 1)^2(m + 2)}{12}\end{aligned}$$

Also solved by Wray G. Brady, Paul S. Bruckman, D. K. Chang, Herta T. Freitag, Northern State College Problems Group, Bob Prielipp, A. G. Shannon, Sahib Singh, Lawrence Somer, Jonathan Weitzman, and the proposer.

Problem Editor's Error

B-438 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let n and w be integers with w odd. Prove or disprove the proposed identity

$$F_{n+2w}F_{n+w} - 2L_wF_{n+w}F_{n-w} + F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

Solution by Paul S. Bruckman, Concord, CA

The identity indicated in B-438 is false. To see this, we only need to set $n = 0$, $w = 1$ in the left member of the proposed identity, which yields, for this expression,

$$F_2F_1 - 2L_1F_1F_{-1} + F_{-1}F_{-2} = 1 \cdot 1 - 2 \cdot 1 \cdot 1 \cdot 1 + 1(-1) = 1 - 2 - 1 = -2;$$

however, in the right member, we obtain $(L_3 - 2L_1)F_0^2 = 0 \neq -2$. This also disposes of B-439.

Disproofs were also given by Herta T. Freitag, Bob Prielipp, Sahib Singh, and the cited proposer. For the proposer's version, see B-464 above.

Companion Problem

B-439 *Proposed by A. P. Hillman, University of New Mexico, Albuquerque, NM*

Can the proposed identity of B-438 be proved by mere verification for a finite set of ordered pairs (n, w)? If so, how few pairs suffice?

Solution by Paul S. Bruckman contained in his solution to B-438.

Converse Does Not Hold

B-440 *Proposed by Jeffrey Shallit, University of California, Berkeley, CA*

(a) Let $n = x^2 + y^2$, with x and y integers not both zero. Prove that there is a nonnegative integer k such that $n \equiv 2^k \pmod{2^{k+2}}$.

(b) If $n \equiv 2^k \pmod{2^{k+2}}$, must n be a sum of squares?

Solution by Paul S. Bruckman, Concord, CA

It is a well-known result of number theory that any positive integer n is representable as the sum of two squares if and only if its prime factorization only contains *even* powers (possibly zero) of primes congruent to 3 (mod 4). In this case, the odd portion of n (n itself, if odd) must be congruent to 1 (mod 4). Thus, $n = 2^k(4s + 1) = s \cdot 2^{k+2} + 2^k$, or $n \equiv 2^k \pmod{2^{k+2}}$. This proves part (a) of the problem, where the desired integer k is simply the greatest power of 2 in the prime factorization of n.

If $n = 21 \cdot 2^k$ $(k \geq 0)$, then $n = 2^k \cdot 3 \cdot 7$, which cannot be a sum of two squares, because of the result quoted above. Nevertheless,

$$n = 2^k(5 \cdot 2^2 + 1) \equiv 2^k \pmod{2^{k+2}}.$$

Hence, the answer to part (b) of the problem is negative. In general, if n contains an even *number* of *odd* powers of primes congruent to 3 (mod 4) in its prime factorization, it will satisfy the given congruence, but cannot be expressed as the sum of two squares.

Also solved by D. K. Chang, M. J. DeLeon, Herta T. Freitag, H. Klauser, Bob Prielipp, Sahib Singh, Lawrence Somer, Gregory Wulczyn, and the proposer.

Sum of Base-b Palindrome Reciprocals

B-441 *Proposed by Jeffrey Shallit, University of California, Berkeley, CA*

A base-b palindrome is a positive integer whose base-b representation reads the same forward and backward. Prove that the sum of the reciprocals of all base-b palindromes converges for any given integer $b \geq 2$.

Solution by H. Klauser, Zurich, Switzerland

Among the $2n$-digit numbers, there are $(b - 1)b^{n-1}$ palindromes. A lower bound for them is b^{2n-1} and the sum of their reciprocals is

$$S_{2n} < (b - 1)b^{n-1}b^{1-2n} = (b - 1)b^{-n}.$$

There are $(b - 1)b^n$ palindromes with $2n + 1$ digits, a lower bound is b^{2n}, and the sum of their reciprocals is

$$S_{2n+1} < (b - 1)b^{-n}.$$

It follows that the sum S of the reciprocals of all palindromes is less than

$$2(b - 1)\sum_{n=1}^{\infty} b^{-n} = 2.$$

Also solved by Wray G. Brady, Paul S. Bruckman, Lawrence Somer, Jonathan Weitzman, and the proposer.

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