4. A. Dalzell. "Book Review of Duckworth's *Structural Patterns . . . .*" *Phoenix, The Canadian Classical Journal* 17 (1963):314-16.
5. G. Duckworth. *Structural Patterns and Proportions in Vergil's Aeneid.* Ann Arbor: University of Michigan Press, 1962.
6. R. Fischler. "Théories mathématiques de la Grande Pyramide." *Crux Mathematicorum* 4 (1978):122-29.
7. R. Fischler. "What Did Herodotus Really Say? or How To Build (a Theory of) the Great Pyramid." *Environment and Planning* 6 (1979):89-93.
8. R. Fischler. "The Early Relationship of Le Corbusier to the 'Golden Number.'" *Environment and Planning B.* 6 (1979):95-103.
9. R. Fischler. "An Investigation of Claims Concerning Seurat and the 'Golden Number.'" To appear in *Gazette des Beaux Arts.*
10. R. Fischler & E. Fischler. "Juan Gris, son milieu et le nombre d'or." *Canadian Art Review* 7 (1980):33-36.
11. R. Fischler. "On Applications of the Golden Ratio in the Visual Arts." *Leonardo* 14 (1981):31-32.
12. G. Hardy & E. Wright. *An Introduction to the Theory of Numbers.* 4th ed. Oxford: Clarendon, 1960.
13. H. Hedian. "The Golden Section and the Artist." *The Fibonacci Quarterly* 14 (1976):406-18,
14. H. Kyburg. *Probability Theory.* Englewood Cliffs, N.J.: Prentice-Hall, 1969.
15. I. Niven & H. Zuckerman. *Introduction to the Theory of Numbers.* New York: Wiley, 1960.
16. H. Shiffman & D. Bobko. "Preference in Linear Partitioning: The Golden Section Reexamined." *Perception and Psychophysics* 24 (1978):102-3.
17. A. Zeising. "Uber die Metamorphosen in den Verhaltnissen der menschilichen Gestalt von der Gerburt bis zur Vollendung des Langenwachstums." *Deutsche Akademie der Naturforscher Nova Acta Leopoldine* 26 (1857):781-879.

*****

EXTENDED BINET FORMS FOR GENERALIZED QUATERNIONS OF HIGHER ORDER

A. L. IAKIN
University of New England, Armidale, Australia

In a prior article [4], the concept of a higher-order quaternion was established and some identities for these quaternions were then obtained. In this paper we introduce a "Binet form" for generalized quaternions and then proceed to develop expressions for extended Binet forms for generalized quaternions of higher order. The extended Binet formulas make possible an approach for generating results which differs from that used in [4].

We recall from Horadam [1] the Binet form for the sequence $W_n(a, b; p, q)$, viz.,

$$W_n = A\alpha^n - B\beta^n$$

where

$$W_0 = a, \qquad W_1 = b$$

$$A = \frac{b - a\beta}{\alpha - \beta}, \qquad B = \frac{b - a\alpha}{\alpha - \beta}$$

and where α and β are the roots of the quadratic equation

$$x^2 - px + q = 0.$$

We define the vectors $\underline{\alpha}$ and $\underline{\beta}$ such that

$$\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3 \quad \text{and} \quad \underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3,$$

where i, j, k are the quaternion vectors as given in Horadam [2].

Now, as in [4], we introduce the operator Ω:

$$\begin{aligned}\Omega W_n &= W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}\\ &= A\alpha^n - B\beta^n + i(A\alpha^{n+1} - B\beta^{n+1}) + j(A\alpha^{n+2} - B\beta^{n+2}) + k(A\alpha^{n+3} - B\beta^{n+3})\\ &= A\alpha^n(1 + i\alpha + j\alpha^2 + k\alpha^3) - B\beta^n(1 + i\beta + j\beta^2 + k\beta^3)\end{aligned}$$

Therefore,

$$\Omega W_n = A\alpha^n\underline{\alpha} - B\beta^n\underline{\beta}. \tag{1}$$

This is the Binet formula for the generalized quaternion of order one. Consider

$$\begin{aligned}\Delta W_n &= W_n + iqW_{n-1} + jq^2W_{n-2} + kq^3W_{n-3}\\ &= A\alpha^n - B\beta^n + iq(A\alpha^{n-1} - B\beta^{n-1}) + jq^2(A\alpha^{n-2} - B\beta^{n-2}) + kq^3(A\alpha^{n-3} - B\beta^{n-3})\\ &= A\alpha^n(1 + iq\alpha^{-1} + jq^2\alpha^{-2} + kq^3\alpha^{-3}) - B\beta^n(1 + iq\beta^{-1} + jq^2\beta^{-2} + kq^3\beta^{-3})\end{aligned}$$

but

$$\alpha\beta = q$$

i.e., $\alpha = q\beta^{-1}$ and $\beta = q\alpha^{-1}$; hence,

$$\Delta W_n = A\alpha^n(1 + i\beta + j\beta^2 + k\beta^3) - B\beta^n(1 + i\alpha + j\alpha^2 + k\alpha^3).$$

Therefore,

$$\Delta W_n = A\alpha^n\underline{\beta} - B\beta^n\underline{\alpha}. \tag{2}$$

Thus we see that the quaternion formed by the Δ operator, that proved so useful in [3] and [4], has a Binet form which is a simple permutation of result (1) above.

We now examine quaternions of order λ (for λ an integer) and prove by induction that

$$\Omega^\lambda W_n = A\alpha^n\underline{\alpha}^\lambda - B\beta^n\underline{\beta}^\lambda. \tag{3}$$

Proof: When $\lambda = 1$, the result is true because

$$\Omega^1 W_n = \Omega W_n = A\alpha^n\underline{\alpha} - B\beta^n\underline{\beta}.$$

Assume that the result is true for $\lambda = m$, i.e.,

$$\Omega^m W_n = A\alpha^n\underline{\alpha}^m - B\beta^n\underline{\beta}^m.$$

Now, for $\lambda = m + 1$,

$$\begin{aligned}\Omega^{m+1}W_n &= \Omega^m W_n + i\Omega^m W_{n+1} + j\Omega^m W_{n+2} + k\Omega^m W_{n+3}\\ &= A\alpha^n\underline{\alpha}^m - B\beta^n\underline{\beta}^m + i(A\alpha^{n+1}\underline{\alpha}^m - B\beta^{n+1}\underline{\beta}^m) + j(A\alpha^{n+2}\underline{\alpha}^m - B\beta^{n+2}\underline{\beta}^m)\\ &\qquad + k(A\alpha^{n+3}\underline{\alpha}^m - B\beta^{n+3}\underline{\beta}^m)\\ &= A\alpha^n(1 + i\alpha + j\alpha^2 + k\alpha^3)\underline{\alpha}^m - B\beta^n(1 + i\beta + j\beta^2 + k\beta^3)\underline{\beta}^m\\ &= A\alpha^n\underline{\alpha}\,\underline{\alpha}^m - B\beta^n\underline{\beta}\,\underline{\beta}^m\\ &= A\alpha^n\underline{\alpha}^{m+1} - B\beta^n\underline{\beta}^{m+1}.\end{aligned}$$

Since the result is true for $\lambda = 1$ and also true for $\lambda = m + 1$ whenever the result holds for $\lambda = m$, it follows from the principle of induction that the result is true for all integral λ. Similarly, it can be shown that

$$\Delta^\lambda W_n = A\alpha^n\underline{\beta}^\lambda - B\beta^n\underline{\alpha}^\lambda. \tag{4}$$

Since

$$\Omega\Delta W_n = \Delta W_n + i\Delta W_{n+1} + j\Delta W_{n+2} + k\Delta W_{n+3}$$

and

$$\Delta\Omega W_n = \Omega W_n + iq\Omega W_{n-1} + jq^2\Omega W_{n-2} + kq^3\Omega W_{n-3},$$

we secure, using equations (2) and (1), respectively,

(5) $$\Omega\Delta W_n = A\alpha^n\underline{\alpha}\,\underline{\beta} - B\beta^n\underline{\beta}\,\underline{\alpha}$$

(6) $$\Delta\Omega W_n = A\alpha^n\underline{\beta}\,\underline{\alpha} - B\beta^n\underline{\alpha}\,\underline{\beta}$$

If we let $\lambda = 2$ in equations (3) and (4) and also use equations (5) and (6), we can derive the six permutations for quaternions of order 3 involving both Ω and Δ operators, namely

(7) $$\Omega^2\Delta W_n = A\alpha^n\underline{\alpha}^2\,\underline{\beta} - B\beta^n\underline{\beta}^2\,\underline{\alpha}$$

(8) $$\Delta^2\Omega W_n = A\alpha^n\underline{\beta}^2\,\underline{\alpha} - B\beta^n\underline{\alpha}^2\,\underline{\beta}$$

(9) $$\Omega\Delta^2 W_n = A\alpha^n\underline{\alpha}\,\underline{\beta}^2 - B\alpha^n\underline{\beta}\,\underline{\alpha}^2$$

(10) $$\Delta\Omega^2 W_n = A\alpha^n\underline{\beta}\,\underline{\alpha}^2 - B\beta^n\underline{\alpha}\,\underline{\beta}^2$$

(11) $$\Omega\Delta\Omega W_n = A\alpha^n\underline{\alpha}\,\underline{\beta}\,\underline{\alpha} - B\beta^n\underline{\beta}\,\underline{\alpha}\,\underline{\beta}$$

(12) $$\Delta\Omega\Delta W_n = A\alpha^n\underline{\beta}\,\underline{\alpha}\,\underline{\beta} - B\beta^n\underline{\alpha}\,\underline{\beta}\,\underline{\alpha}$$

We now pause to investigate the effects of operators Ω^* and Δ^* on the Binet forms. Note from [4] that

$$\Omega^*\Delta W_n = \Delta W_n + \Delta W_{n+1}\cdot i + \Delta W_{n+2}\cdot j + q^3\Delta W_{n+3}\cdot k = \Delta\Omega W_n$$

and

$$\Delta^*\Omega W_n = \Omega W_n + q\Omega W_{n-1}\cdot i + q^2\Omega W_{n-2}\cdot j + q^3\Omega W_{n-3}\cdot k = \Omega\Delta W_n$$

and thus the operators Ω^* and Δ^* provide no new results for quaternions of order 2. Since equations (7) to (12) and equations (3) and (4) for $\lambda = 3$ provide every possible triad of combination of $\underline{\alpha}$ and $\underline{\beta}$, it is unlikely that quaternions of order 3 involving the starred operators will produce any Binet form distinct from those given. A close inspection of the *modus operandi* of Ω^* and Δ^* verifies that this is indeed the case. For example, it is easily calculated that

$$\Omega^*\Delta\Omega W_n = A\alpha^n\underline{\beta}\,\underline{\alpha}^2 - B\beta^n\underline{\alpha}\,\underline{\beta}^2$$

which is the same expression as that for $\Delta\Omega^2 W_n$.

We can generalize these statements to say that the operators Ω^* and Δ^* yield no results that cannot be obtained solely by manipulating the operators Ω and Δ.

From equations (3) and (4), it can be readily shown that, for μ an integer,

(13) $$\Delta^\lambda\Omega^\mu W_n = A\alpha^n\underline{\beta}^\lambda\,\underline{\alpha}^\mu - B\beta^n\underline{\alpha}^\lambda\,\underline{\beta}^\mu$$

(14) $$\Omega^\lambda\Delta^\mu W_n = A\alpha^n\underline{\alpha}^\lambda\,\underline{\beta}^\mu - B\beta^n\underline{\beta}^\lambda\,\underline{\alpha}^\mu$$

The pattern between the higher-order quaternions and their related Binet forms being clearly established, we deduce, for integral λ_i, $i = 1, \ldots, m$, the ensuing extended Binet formulas of finite order:

(15) $$\Omega^{\lambda_1}\Delta^{\lambda_2}\ldots\Omega^{\lambda_m}W_n = A\alpha^n\underline{\alpha}^{\lambda_1}\underline{\beta}^{\lambda_2}\ldots\underline{\alpha}^{\lambda_m} - B\beta^n\underline{\beta}^{\lambda_1}\underline{\alpha}^{\lambda_2}\ldots\underline{\beta}^{\lambda_m}$$

(16) $$\Omega^{\lambda_1}\Delta^{\lambda_2}\ldots\Delta^{\lambda_m}W_n = A\alpha^n\underline{\alpha}^{\lambda_1}\underline{\beta}^{\lambda_2}\ldots\underline{\beta}^{\lambda_m} - B\beta^n\underline{\beta}^{\lambda_1}\underline{\alpha}^{\lambda_2}\ldots\underline{\alpha}^{\lambda_m}$$

(17) $$\Delta^{\lambda_1}\Omega^{\lambda_2}\ldots\Omega^{\lambda_m}W_n = A\alpha^n\underline{\beta}^{\lambda_1}\underline{\alpha}^{\lambda_2}\ldots\underline{\alpha}^{\lambda_m} - B\beta^n\underline{\alpha}^{\lambda_1}\underline{\beta}^{\lambda_2}\ldots\underline{\beta}^{\lambda_m}$$

(18) $$\Delta^{\lambda_1}\Omega^{\lambda_2}\ldots\Delta^{\lambda_m}W_n = A\alpha^n\underline{\beta}^{\lambda_1}\underline{\alpha}^{\lambda_2}\ldots\underline{\beta}^{\lambda_m} - B\beta^n\underline{\alpha}^{\lambda_1}\underline{\beta}^{\lambda_2}\ldots\underline{\alpha}^{\lambda_m}$$

From equations (2.6) and (2.7) of Horadam [1], we derive the following Binet formulas:

$$\Omega^\lambda U_n = [\alpha^{n+1}\underline{\alpha}^\lambda - \beta^{n+1}\underline{\beta}^\lambda]/d \tag{19}$$

$$\Delta^\lambda U_n = [\alpha^{n+1}\underline{\beta}^\lambda - \beta^{n+1}\underline{\alpha}^\lambda]/d \tag{20}$$

$$\Omega^\lambda V_n = \alpha^n\underline{\alpha}^\lambda + \beta^n\underline{\beta}^\lambda \tag{21}$$

$$\Delta^\lambda V_n = \alpha^n\underline{\beta}^\lambda + \beta^n\underline{\alpha}^\lambda \tag{22}$$

We now use the extended Binet formulas to establish some identities. As an example, consider a simple generalization of equation (28) in [4]:

$$\begin{aligned}\Omega^\lambda V_r\Omega^\mu W_n &= (\alpha^r\underline{\alpha}^\lambda + \beta^r\underline{\beta}^\lambda)(A\alpha^n\underline{\alpha}^\mu - B\beta^n\underline{\beta}^\mu)\\ &= A\alpha^{n+r}\underline{\alpha}^{\lambda+\mu} + A\alpha^n\beta^r\underline{\beta}^\lambda\underline{\alpha}^\mu - B\alpha^r\beta^n\underline{\alpha}^\lambda\underline{\beta}^\mu - B\beta^{n+r}\underline{\beta}^{\lambda+\mu}\\ &= A\alpha^{n+r}\underline{\alpha}^{\lambda+\mu} - B\beta^{n+r}\underline{\beta}^{\lambda+\mu} + \alpha^r\beta^r(A\alpha^{n-r}\underline{\beta}^\lambda\underline{\alpha}^\mu - B\beta^{n-r}\underline{\alpha}^\lambda\underline{\beta}^\mu)\end{aligned}$$

Therefore,

$$\Omega^\lambda V_r\Omega^\mu W_n = \Omega^{\lambda+\mu}W_{n+r} + q^r\Delta^\lambda\Omega^\mu W_{n-r} \tag{23}$$

This, in turn, can easily be further extended to provide a most generalized formula, viz.,

$$\begin{aligned}&\Omega^{\lambda_1}\Delta^{\lambda_2}\ \ldots\ \Omega^{\lambda_m}V_r\Omega^{\mu_1}\Delta^{\mu_2}\ \ldots\ \Omega^{\mu_m}W_n \\ &= \Omega^{\lambda_1}\Delta^{\lambda_2}\ \ldots\ \Omega^{\lambda_m+\mu_1}\Delta^{\mu_2}\ \ldots\ \Omega^{\mu_m}W_{n+r} + q^r\Delta^{\lambda_1}\Omega^{\lambda_2}\ \ldots\ \Delta^{\lambda_m}\Omega^{\mu_1}\Delta^{\mu_2}\ \ldots\ \Omega^{\mu_m}W_{n-r}\end{aligned} \tag{24}$$

It is obvious to the reader that other similar generalizations of the results in [4] can be procured by this method.

We now look at an equation not contained in [3] or [4]. Consider

$$\begin{aligned}\left(\frac{\Omega^\lambda V_n + d\Omega^\lambda U_{n-1}}{2}\right)^m &= \left(\frac{\alpha^n\underline{\alpha}^\lambda + \beta^n\underline{\beta}^\lambda + \alpha^n\underline{\alpha}^\lambda - \beta^n\underline{\beta}^\lambda}{2}\right)^m = (\alpha^n\underline{\alpha}^\lambda)^m = \alpha^{mn}\underline{\alpha}^{\lambda m}\\ &= \frac{2\alpha^{mn}\underline{\alpha}^{\lambda m}}{2} = \frac{\alpha^{mn}\underline{\alpha}^{\lambda m} + \beta^{mn}\underline{\beta}^{\lambda m} - \beta^{mn}\underline{\beta}^{\lambda m} + \alpha^{mn}\underline{\alpha}^{\lambda m}}{2}\end{aligned}$$

Thus

$$\left(\frac{\Omega^\lambda V_n + d\Omega^\lambda U_{n-1}}{2}\right)^m = \frac{\Omega^{\lambda m}V_{mn} + d\Omega^{\lambda m}U_{mn-1}}{2} \tag{25}$$

This is a De Moivre type identity for higher-order quaternions.

Thus we see that the extended Binet formulas not only permit direct verification of the identities contained in [3] and [4], and extensions of these as we have shown, but also facilitate the attainment of new results.

REFERENCES

1. A. F. Horadam. "Basic Properties of a Certain Generalized Sequence of Numbers." *The Fibonacci Quarterly* 3 (1965):161-75.
2. A. F. Horadam. "Complex Fibonacci Numbers and Fibonacci Quaternions." *Amer. Math. Monthly* 70 (1963):289-91.
3. A. L. Iakin. "Generalized Quaternions with Quaternion Components." *The Fibonacci Quarterly* 15 (1977):350-52.
4. A. L. Iakin. "Generalized Quaternions of Higher Order." *The Fibonacci Quarterly* 15 (1977):343-46.

*****