ALMOST ARITHMETIC SEQUENCES AND COMPLEMENTARY SYSTEMS

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What about the sequence 3, 6, 9, 12, 15, ... ? If this is simply the arithmetic sequence $\{3n\}$, then its study would be essentially that of the positive integers. However, suppose the nth term is $[(3 + 1/\sqrt{29})n]$, or perhaps $[(4 - 5/\sqrt{37})n]$, where $[x]$ means the greatest integer $\leq x$. In these sequences, 15 is followed by 19 rather than 18. Such *almost arithmetic* sequences have many interesting properties which have been discovered only in recent years. Of special interest are complementary systems of such sequences.

The adjective "complementary" means that every positive integer occurs exactly once in exactly one of a given set of sequences. Consider, for example, the three sequences

(1) 1, 4, 6, 8, 10, 13, ...; 2, 5, 9, 12, 16, ...; 3, 7, 11, 14, ...;

which can be accounted for as follows: If the positive integers that are squares, twice squares, or thrice squares are all arranged in increasing order, we find at the beginning

(2) 1, 2, 3, 4, 8, 9, 12, 16, 18, 25, 27, 32, 36.

Each of these numbers occupies a position in the arrangement. In particular, the squares 1, 4, 9, 16, 25, 36, ... occupy positions numbered 1, 4, 6, 8, 10, 13, ..., the first sequence in (1). This line of reasoning can be extended to show that the three sequences in (1) are given, respectively, by the formulas

$$(1') \qquad n + \left[\frac{n}{\sqrt{2}}\right] + \left[\frac{n}{\sqrt{3}}\right],\ [n\sqrt{2}] + n + \left[\frac{n\sqrt{2}}{\sqrt{3}}\right],\ [n\sqrt{3}] + \left[\frac{n\sqrt{3}}{\sqrt{2}}\right] + n.$$

The three sequences in (1) may be compared with the sequences

1, 4, 7, 10, 13, 16, ...; 2, 5, 8, 11, 14, ...; 3, 6, 9, 12, ...;

which form a complementary system of arithmetic sequences given by $3n + 1$, $3n + 2$, and $3n + 3$. Each has a common difference, or *slope*, equal to 3. Similarly, the sequences in (1) have slopes $s = 1 + 1/\sqrt{2} + 1/\sqrt{3}$, $\sqrt{2}s$, and $\sqrt{3}s$, as shown by formulas (1′). Here the similarity ends, however. Writing $a_n = 3n + 1$, we call to mind the very simple recurrence relation $a_{n+1} - a_n = 3$. On the other hand, writing $b_n = n + [n/\sqrt{2}] + [n/\sqrt{3}]$, we find $b_{n+1} - b_n \neq s$ for all n. Instead, $b_{n+1} - b_n$ takes values 1, 2, and 3, depending on n. Moreover, $a_{n+2} - a_n = 6$ for all n, whereas $b_{n+2} - b_n$ takes values 4, 5, and 6.

We are now in a position to state the purpose of this note: first, to introduce a definition of "almost arithmetic" that covers sequences as in (1), and then to present some theorems about almost arithmetic sequences and complementary systems.

One more thought before defining the general almost arithmetic sequence $\{a_n\}$ is that there should be a real number u such that a_n must stay close to the arithmetic sequence nu. Specifically, $a_n - nu$ should stay bounded as n goes through the positive integers, and this could be used as the defining property for "almost arithmetic" sequences. However, this property depends on the existence of a real number u, and since the a_n are positive integers, a definition which refers only to positive integers is much to be preferred. From such a definition, we should be able to determine the number u. The following definition meets these requirements.

Suppose $\ell \le k$ are nonnegative integers and $\{a_n\}$ is a strictly increasing sequence of positive integers satisfying

$$0 \le a_{m+n} - a_m - a_n + \ell \le k, \text{ for all } m, n \ge 1. \tag{3}$$

The sequence $\{a_n\}$ is *almost arithmetic*, or, more specifically, (k, ℓ)*-arithmetic*.

It is fairly easy to check that for any positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ and $\beta_1, \beta_2, \ldots, \beta_k$ the sequence with nth term

$$a_n = [\alpha_1 n] + [\alpha_2 n] + \cdots + [\alpha_k n] \tag{4}$$

is $(k, 0)$-arithmetic, and the sequence with nth term

$$a_n = [\alpha_1 n + \beta_1] + [\alpha_2 n + \beta_2] + \cdots + [\alpha_k n + \beta_k] \tag{5}$$

is (k', ℓ)-arithmetic for some ℓ and some $k' \ge k$.

For example, the sequence $\{3n\}$ is $(0, 0)$-arithmetic; $\{3n+1\}$ is $(0, 1)$-arithmetic, and $\{n + [n/\sqrt{2}] + [n/\sqrt{3}]\}$ is $(2, 0)$-arithmetic.

As we shall soon see, there are many almost arithmetic sequences $\{a_n\}$ for which no formula in closed form for a_n is known. Nevertheless, our first theorem will show that every almost arithmetic sequence $\{a_n\}$ must have a slope u, and a_n must stay close to nu.

Theorem 1: If $\{a_n\}$ is a (k, ℓ)-arithmetic sequence, then the number $u = \lim_{n\to\infty} \frac{a_n}{n}$, hereinafter referred to as the *slope* of $\{a_n\}$, exists, and

$$a_n \le nu + \ell \le a_n + k, \text{ for } n = 1, 2, \ldots . \tag{6}$$

Proof: Let $\varepsilon > 0$, and let m be so large that

$$\max\left\{\frac{\ell}{m}, \frac{k-\ell}{m}\right\} < \varepsilon.$$

for any $n > m$, we have $n = qm + r$ where $q = [n/m]$ and $0 \le r < m$. By (3),

$$a_m - \ell \le a_n - a_{n-m} \le a_m - \ell + k$$

and

$$a_m - \ell \le a_{n-m} - a_{n-2m} \le a_m - \ell + k$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

and finally $$a_m - \ell \le a_{n-(q-1)m} - a_r \le a_m - \ell + k.$$

Adding these:

$$q(a_m - \ell) \le a_n - a_r \le q(a_m - \ell + k).$$

Now adding $a_r - qa_m$ and dividing by n yields

$$\frac{a_r}{n} - \frac{\ell q}{n} \le \frac{a_n}{n} - \frac{qa_m}{n} \le \frac{q}{n}(k - \ell) + \frac{a_r}{n}.$$

When this is added to the easily verified

$$-\frac{a_m}{n} \le \frac{qa_m}{n} - \frac{a_m}{m} \le 0,$$

one obtains

$$\frac{a_r - a_m}{n} - \varepsilon < \frac{a_r - a_m}{n} - \frac{\ell}{m} \le \frac{a_r - a_m - \ell q}{n} \le \frac{a_n}{n} - \frac{a_m}{m} \le \frac{q(k - \ell) + a_r}{n}$$

$$\le \frac{k - \ell}{m} + \frac{a_r}{n} < \varepsilon + \frac{a_r}{n}.$$

As $n \to \infty$ we see that $\left|\frac{a_n}{n} - \frac{a_m}{m}\right| \le \varepsilon$, so that $\left\{\frac{a_n}{n}\right\}$, as a Cauchy sequence, converges.

Now as a first step in an induction argument,

$$a_n - \ell \le a_{2n} - a_n \le a_n - \ell + k.$$

Assume for arbitrary $j > 3$ that

$$(j - 2)(a_n - \ell) \le a_{(j-1)n} - a_n \le (j - 2)(a_n - \ell + k).$$

Adding this with $a_n - \ell \le a_{jn} - a_{(j-1)n} \le a_n - \ell + k$ gives

$$(j - 1)(a_n - \ell) \le a_{jn} - a_n \le (j - 1)(a_n - \ell + k),$$

which concludes the induction argument. This set of inequalities is equivalent to

$$ja_n - (j - 1)\ell \le a_{jn} \le ja_n + (j - 1)(k - \ell).$$

Dividing by jn,

$$\frac{a_n}{n} - \frac{1}{n}\frac{j-1}{j}\ell \le \frac{a_{jn}}{jn} \le \frac{a_n}{n} + \frac{1}{n}\frac{j-1}{j}(k - \ell).$$

Since $\lim\limits_{j\to\infty} \frac{a_{jn}}{jn} = u$, we have

$$\frac{a_n}{n} - \frac{\ell}{n} \le u \le \frac{a_n}{n} + \frac{1}{n}(k - \ell),$$

and (6) follows.

Theorem 1 should be compared with similar results in Pólya and Szegö [7, pp. 23-24].

Note the contrast between the defining inequality (3) and Theorem 1. The former is entirely combinatorial, whereas the notion of slope is analytic. Specifically, when ℓ is the *least* integer such that

$$a_{m+n} - a_m - a_n + \ell \ge 0, \text{ for all } m, n \ge 1,$$

and if k is the *least* integer such that

$$a_{m+n} - a_m - a_n + \ell \le k, \text{ for all } m, n \ge 1,$$

then k counts the extent to which the sequence $\{a_n - \ell\}$ deviates from the rule

$$c_{m+n} - c_m - c_n = 0;$$

that is, from being an arithmetic sequence. On the other hand, the slope u gives the average growth rate of $\{a_n\}$. With this analytic notion in mind, we may predict that if $\{a_n\}$ has slope u and $\{b_n\}$ slope v, then the composite sequences $\{a_{b_n}\}$ and $\{b_{a_n}\}$ will have slope uv. Or, if the given sequences are disjoint, we can combine them in increasing order, thus getting a sequence with slope $(u^{-1} + v^{-1})^{-1}$, the harmonic mean of u and v. Then returning to a combinatorial attitude, we may ask about the bounding numbers k and ℓ for these new sequences. Our first theorem of the sort just suggested shows how to make almost arithmetic sequences from a given real $u \ge 1$.

Theorem 2: If $u \ge 1$ is a real number and $\{a_n\}$ is an increasing sequence of positive integers satisfying $0 \le nu - a_n + \ell \le k$ for $0 \le \ell \le k$ and for $n = 1, 2, \ldots$, then $\{a_n\}$ is a $(3k, k + \ell)$-arithmetic sequence with slope u.

Proof: Subtracting $0 \le (m+n)u - a_{m+n} + \ell \le k$ from $0 \le mu - a_m + \ell \le k$ gives $-k \le a_{m+n} - a_m - nu \le k$. This implies $nu \le a_{m+n} - a_m + k \le nu + 2k$. Bounds for nu come from $0 \le nu - a_n + \ell \le k$, namely $a_n - \ell \le nu \le a_n - \ell + k$. Thus

$$a_n - \ell \le a_{m+n} - a_m + k \le a_n - \ell + 3k,$$

or equivalently,

$$0 \le a_{m+n} - a_m - a_n + \ell + k \le 3k,$$

as required.

As an example, let $a_n = 2n$ if n is prime and $2n+1$ otherwise. Then $k = \ell = 1$ in Theorem 2, and $\{a_n\}$ is a (3, 2)-arithmetic sequence. Actually, $\{a_n\}$ is also a (2, 2)-arithmetic sequence, which is saying more. This example shows that the k and ℓ in Theorem 2 need not be the *least* values for which (3) holds. This same observation holds for the theorems that follow.

Consider next $a_n = 10n + 2$ and $b_n = 10 \quad + 5$ for $n = 0, 1, 2, \ldots$. We combine these to form the sequence $\{c_n\}$ given by 2, 5, 12, 15, 22, 25, ..., and ask if this is an almost arithmetic sequence. If so, what numbers k, ℓ describe the maximal spread which c_n has away from $5n$? The question leads to the following theorem about disjoint unions of almost arithmetic sequences.

Theorem 3: Suppose $\{a_n\}$ is a (k, ℓ)-arithmetic sequence and $\{b_n\}$ is a (k', ℓ')-arithmetic sequence, disjoint from $\{a_n\}$ in the sense that $b_n \neq a_m$ for all m and n. Let $\{c_n\}$ be the union of $\{a_n\}$ and $\{b_n\}$. Then $\{c_n\}$ is a $(\mathcal{K}, \mathcal{L})$-arithmetic sequence for some $\mathcal{K}$ and $\mathcal{L}$ (given in the proof). If $\{a_n\}$ has slope u and $\{b_n\}$ has slope v, then $\{c_n\}$ has slope $(u^{-1} + v^{-1})^{-1}$.

Proof: Let n be a positive integer.

Case 1. Suppose $c_n = a_N$ for some N. Let $x = Nu/v$. By Theorem 1,

$$xv - k + \ell \leq a_N \leq xv + \ell \quad \text{and} \quad iv - k' + \ell' \leq b_i \leq iv + \ell', \; i = 1, 2, \ldots .$$

The inequality $iv + \ell' \leq xv - k + \ell$ shows that $b_i \leq a_N$ whenever

$$i \leq x + (\ell - \ell' - k)/v.$$

Similarly, $b_i \geq a_N$ for

$$i \geq x + (\ell - \ell' + k')/v.$$

Thus, the number of b_i which are $\leq a_N$ is $x + \delta$, where

$$(\ell - \ell' - k)/v \leq \delta \leq (\ell - \ell' + k')/v,$$

so that

$$n = (\#a_i \leq a_N) + (\#b_i \leq a_N) = N + Nu/v + \delta.$$

Multiplying by $w = uv/(u + v)$ gives $Nu = (n - \delta)w$. Now, substituting this and $a_N = c_n$ into $Nu - k + \ell \leq a_N \leq Nu + \ell$, we obtain

$$(n - \delta)w - k + \ell \leq c_n \leq (n - \delta)w + \ell.$$

Case 2. Suppose $c_n = b_N$ for some N. As in Case 1, there exists δ' satisfying $(\ell' - \ell - k')/u \leq \delta' \leq (\ell' - \ell + k)/u$ such that

$$(n - \delta')w - k' + \ell' \leq c_n \leq (n - \delta')w + \ell'.$$

To accommodate both cases, let

$$\mathcal{L}'' = \min \begin{cases} \ell - k - \delta w \\ \ell' - k' - \delta' w \end{cases} \quad \text{and} \quad \mathcal{K}'' = \max \begin{cases} \ell - \delta w \\ \ell' - \delta' w, \end{cases}$$

and then let

$$\mathcal{L}' = \begin{cases} \mathcal{L}'' \text{ if } \mathcal{L}'' \text{ is an integer} \\ [\mathcal{L}''] + 1 \text{ otherwise} \end{cases} \quad \text{and} \quad \mathcal{K}' = [\mathcal{K}''].$$

Now $nw + \mathcal{L}' \leq c_n \leq nw + \mathcal{K}'$, so that $0 \leq nw - c_n + \mathcal{K}' \leq \mathcal{K}' - \mathcal{L}'$. By Theorem 2, $\{c_n\}$ is $(\mathcal{K}, \mathcal{L})$-arithmetic, where $\mathcal{K} = 3(\mathcal{K}' - \mathcal{L}')$ and $\mathcal{L} = 2\mathcal{K}' - \mathcal{L}'$.

The theorem just proved has an interesting application to complementary systems, as follows.

Theorem 4: Suppose $\{a_{1n}\}, \{a_{2n}\}, \ldots, \{a_{mn}\}$ are almost arithmetic sequences that comprise a complementary system. Let $u_i = \lim_{n\to\infty} \dfrac{a_{in}}{n}$ for $i = 1, 2, \ldots, m$. Then

$$\frac{1}{u_1} + \frac{1}{u_2} + \cdots + \frac{1}{u_m} = 1.$$

Proof: As members of a complementary system, $\{a_{1n}\}$ and $\{a_{2n}\}$ are disjoint. By Theorem 3, their union is an almost arithmetic sequence with slope w satisfying $1/w = 1/u_1 + 1/u_2$. Assume for arbitrary $k \le m - 1$ that the union

$$\{a_{1n}\} \cup \{a_{2n}\} \cup \cdots \cup \{a_{kn}\}$$

is almost arithmetic with slope u satisfying $1/u = 1/u_1 + \cdots + 1/u_k$. Theorem 3 applies. By mathematical induction on k, we have $1/v = 1/u_1 + \cdots + 1/u_m$, where v is the slope of the union of all the given sequences, that is, 1.

In case $m = 2$, the identity

$$1 = \sum_{i=1}^{m} 1/u_i$$

is the subject of the famous Beatty Problem [1] of 1926. An extensive bibliography on results stemming from Beatty's Problem and other research on sequences of the form $\{[un]\}$ is given in Stolarsky [8]; the interested reader should also consult Fraenkel, Mushkin, and Tassa [3]. A generalization of Beatty's Problem by Skolem [7] is that sequences $\{[un]\}$ and $\{[vn]\}$, where u and v are positive irrationals, are disjoint if and only if $a/u + b/v = 1$ for some integers a and b. Skolem's generalization suggests a still more general question, which we state here hoping that an answer will someday by found: What criteria exist for disjointness of two sequences of the form (4), for $k \ge 2$?

We turn next to composites of almost arithmetic sequences.

Theorem 5: Composites of almost arithmetic sequences are almost arithmetic. Specifically, if $\{a_n\}$ is (k, ℓ)-arithmetic with slope u and $\{b_n\}$ is (k', ℓ')-arithmetic with slope v, then the sequence $\{c_n\}$ defined by $c_n = b_{a_n}$ is $(b_\ell + b_{k-\ell} + 3k' - 2\ell', b_\ell + k')$-arithmetic with slope uv. (Here $b_0 \equiv 0$.)

Proof: We must show that

$$0 \le c_{m+n} - c_m - c_n + b_\ell + k' \tag{7}$$

and

$$c_{m+n} - c_m - c_n + b_\ell + k' \le b_\ell - b_{k-\ell} + 3k' - 2\ell'. \tag{8}$$

Now

$$\begin{aligned} 0 &\le b_{a_m + a_n} - b_{a_m} - b_{a_n} + \ell' \text{ by (3)} \\ &\le b_{a_{m+n} + \ell} - b_{a_m} - b_{a_n} + \ell' \text{ since } a_m + a_n \le a_{m+n} + \ell \\ &\le (b_{a_{m+n}} + b_\ell + k' - \ell') - b_{a_m} - b_{a_n} + \ell' \text{ by (3).} \end{aligned}$$

This proves (7). To prove (8),

$$\begin{aligned} b_{a_{m+n}} - b_{a_m} - b_{a_n} + b_\ell + k' &\le b_{a_m + a_n + k + \ell} - b_{a_m} - b_{a_n} + b_\ell + k' \\ &\le b_{a_m + a_n} + b_{k-\ell} + k' - \ell' - b_{a_m} - b_{a_n} + b_\ell + k' \\ &\le b_{a_m} + b_{a_n} + k' - \ell' + b_{k-\ell} + k' - \ell' \\ &\qquad - b_{a_m} - b_{a_n} + b_\ell + k' \\ &= b_\ell + b_{k-\ell} + 3k' - 2\ell', \end{aligned}$$

as required.

For slopes we have $a_n \sim un$ and $b_n \sim vn$, where the symbol $\sim$ abbreviates the relationship indicated in (6). Consequently, $b_{a_n} \sim va_n \sim vun$.

To illustrate Theorem 5, let $a_n = [\sqrt{2}n]$ and $b_n = [\sqrt{3}n]$. Each provides a (1, 0)-arithmetic sequence. The composite $b_{a_n} = [\sqrt{3}[\sqrt{2}n]]$ has slope $\sqrt{6}$ and is (4, 1)-arithmetic. The same is true for $a_{b_n} = [\sqrt{2}[\sqrt{3}n]]$.

Theorem 6: The complement of a (k, ℓ)-arithmetic sequence $\{a_n\}$ having slope $u > 1$ is a $\left(\left[\frac{3(u+k)}{u-1}\right], \left[\frac{u+2k-\ell}{u-1}\right]\right)$-arithmetic sequence with slope $u/(u-1)$.

Proof: The complement of $\{a_n\}$ is the increasing sequence $\{a_n^*\}$ of all positive integers missing from $\{a_n\}$. By (6) we can write

$$a_n = nu + \delta, \text{ where } \ell - k \le \delta = \delta(n) \le \ell.$$

Then the inequality $a_i < a_n^*$ can be expressed as $i < (a_n^* - \delta)/u$, and the greatest such i is $[(a_n^* - \delta)/u]$. Now $a_n^* = n + f(a_n^*)$, where $f(x)$ is the number of terms a_i satisfying $a_i < x$. Thus $a_n^* = n + [(a_n^* - \delta)/u]$, and

$$n + (a_n^* - \delta)/u - 1 \le a_n^* \le n + (a_n^* - \delta)/u.$$

This readily leads to

$$\delta \le un - (u-1)a_n^* \le u + \delta,$$

so that

$$0 \le \frac{un}{u-1} - a_n^* + \frac{k-\ell}{u-1} \le \frac{u+k}{u-1},$$

and we conclude, by the method of proof of Theorem 2, that $\{a_n^*\}$ is an almost arithmetic sequence of the required sort.

Theorem 6 shows, for example, that the set of *all* positive integers not of the form $[\sqrt{7}n + \sqrt{3}] + [\sqrt{7}n - \sqrt{3}] = a_n$ forms an almost arithmetic sequence. Suppose that, given a sequence such as $\{a_n\}$, we remove a *sub*sequence which is almost arithmetic, for example $\{a_{[\sqrt{7}n]}\}$. Will the remaining terms of $\{a_n\}$ still form an almost arithmetic sequence? We call such remaining terms the *relative complement* (of $\{a_{[\sqrt{7}n]}\}$ in $\{a_n\}$), and have the following strengthening of Theorem 6.

Theorem 7: The relative complement of an almost arithmetic subsequence of an almost arithmetic sequence is almost arithmetic.

Proof: Suppose $\{a_{n_i}\}$ is an almost arithmetic subsequence of an almost arithmetic sequence $\{a_n\}$. By Theorem 1, there exist positive real u and v and nonnegative integers ℓ, k, ℓ', k' such that

(9) $$a_{n_i} \le n_i u + \ell \le a_{n_i} + k, \quad \ell \le k, \quad i = 1, 2, \ldots,$$

and

(10) $$a_{n_i} \le iv + \ell' \le a_{n_i} + k', \quad \ell' \le k', \quad i = 1, 2, \ldots .$$

Dividing by u in (9) and (10) leads to

$$n_i + \frac{\ell}{u} - \frac{k}{u} \le \frac{a_{n_i}}{u} \le i\frac{v}{u} + \frac{\ell'}{u} \le \frac{a_{n_i}}{u} + \frac{k'}{u} \le n_i + \frac{\ell}{u} + \frac{k'}{u},$$

so that

$$0 \le i\frac{v}{u} - n_i + \frac{\ell' + k - \ell}{u} \le \frac{k + k'}{u}.$$

Thus, by Theorem 2, the sequence $\{n_i\}$ is almost arithmetic. By Theorem 7, the complementary sequence $\{n_i^*\}$, consisting of all positive integers which are not terms of $\{n_i\}$, is almost arithmetic. By Theorem 6, the sequence $\{a_{n_i^*}\}$, which consists of all the a_n's missing from $\{a_{n_i}\}$, is almost arithmetic, as was to be proved.

Corollary to the Proof of Theorem 7: Suppose $\{a_{n_i}\}$ is an almost arithmetic subsequence of an almost arithmetic sequence $\{a_n\}$. Then the sequence $\{n_i\}$ is almost arithmetic.

We now return to the complementary system

(1) 1, 4, 6, 8, 10, 13, ...; 2, 5, 9, 12, 16, ...; 3, 7, 11, 14, ...

Writing these sequences as $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, we list all the positive integers as follows:

$$a_1, b_1, c_1, a_2, b_2, a_3, c_2, a_4, b_3, a_5, c_3, \ldots .$$

Removing all the c_i leaves

(1″) $$a_1, b_1, a_2, b_2, a_3, a_4, b_3, a_5, \ldots .$$

Now let $\{a_n\} \oplus \{c_n\}$ and $\{b_n\} \oplus \{c_n\}$ represent, respectively, the number of the position of a_n and b_n in (1″), counting from the left. These two sequences form a complementary system of almost arithmetic sequences. In fact, for comparison with formulas (1′), one may easily check that

$$\{a_n\} \oplus \{c_n\} = \left\{n + \left[\frac{n}{\sqrt{2}}\right]\right\} = \{1, 3, 5, 6, 8, 10, 11, 13, 15, 17, 18, 20, \ldots\}$$

$$\{b_n\} \oplus \{c_n\} = \{n + [\sqrt{2}n] = \{2, 4, 7, 9, 12, 14, 16, 19, 21, \ldots\}.$$

We define $\oplus$ in general as follows: For *disjoint* strictly increasing sequences $\{a_n\}$ and $\{c_n\}$ of positive integers, let $\{d_n\}$ be the sequence obtained by writing all the a_i and a_i^* in increasing order and then removing all the c_i. Then

$$\{a_n\} \oplus \{c_n\}$$

is the sequence whose nth term is the position of a_n in the sequence $\{d_n\}$.

Even if $\{a_n\}$ and $\{c_n\}$ are not disjoint, we define a second operation $\ominus$ as follows: Construct a sequence $\{e_n\}$ by putting c_n at position c_n for all n and filling all the remaining positions with the a_i and a_i^* written in increasing order. Then

$$\{a_n\} \ominus \{c_n\}$$

is the sequence whose nth term is the position of a_n in the sequence $\{e_n\}$.

One relationship between $\oplus$ and $\ominus$ is indicated by the identity

$$(\{a_n\} \ominus \{c_n\}) \oplus \{c_n\} = \{a_n\}.$$

Also,

$$(\{a_n\} \ominus \{c_n\}) \ominus \{c_n\} = \{a_n\}$$

in case $\{a_n\}$ and $\{c_n\}$ are disjoint.

Both operations $\oplus$ and $\ominus$ can be used on any given complementary system of sequences $\{a_{1n}\}$, $\{a_{2n}\}$, ..., $\{a_{mn}\}$, $m \geq 2$, to produce new complementary systems whose sequences remain almost arithmetic in case the original sequences were so, as we shall see in Theorems 8 and 9. Specifically,

$$\{a_{1n}\} \oplus \{a_{mn}\}, \ldots, \{a_{m-1,n}\} \oplus \{a_{mn}\}$$

is a complementary system of $m - 1$ sequences, and for any strictly increasing sequence $\{c_n\}$ of positive integers, the collection

$$\{a_{1n}\} \ominus \{c_n\}, \ldots, \{a_{mn}\} \ominus \{c_n\},$$

together with $\{c_n\}$ itself, is a complementary system of $m + 1$ sequences.

What about slopes and formulas for the nth terms of sequences arising from $\oplus$ and $\ominus$? We have the following two theorems.

Theorem 8: Suppose $\{a_n\}$ and $\{b_n\}$ are disjoint almost arithemtic sequences having slopes u and v, respectively. Let $c_n = b_n + n - 1$, then

$$\{a_n\} \oplus \{b_n\} = \{2a_n \ominus c^*_{a_n}\}$$

is an almost arithmetic sequence having slope $u - u/v$.

Proof: Let $(\#b_i \le n)$ denote the number of b_i that are $\le n$. Using the formula $CF(n) = n + f^{\dagger}(n)$ on p. 457 of Lambek and Moser [6], we find

$$n + (\#b_i \le n) = n\text{th positive integer not in the sequence } \{b_n + n - 1\},$$

so that

$$(\#b_i \le a_n) = -a_n + a_n\text{th term of the complement of } \{b_n + n - 1\},$$

whence the nth term of $\{a_n\} \oplus \{b_n\}$, which is clearly $a_n - (\#b_i \le a_n)$, must equal $2a_n - c^*_{a_n}$. Since $\{c_n\}$ is almost arithmetic with slope $v+1$, $\{c^*_n\}$ is almost arithmetic with slope $1+1/v$, by Theorem 6. Then $\{c^*_{a_n}\}$ is almost arithmetic with slope $u(1+1/v)$, by Theorem 5. Thus, $\{2a_n - c^*_{a_n}\}$ is almost arithmetic with slope $2u - u(1+1/v)$.

Theorem 9: Suppose $\{a_n\}$ and $\{b_n\}$ are almost arithmetic sequences having slopes u and v, respectively. Then

$$\{a_n\} \ominus \{b_n\} = \{b^*_{a_n}\}$$

is an almost arithmetic sequence with slope $uv/(v-1)$.

Proof: By definition, the nth term of $\{a_n\} \ominus \{b_n\}$ is the a_nth positive integer not one of the b_i, as claimed. As a composite of a complement, this is an almost arithmetic sequence with slope $uv/(v-1)$, much as in the proof of Theorem 8.

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SUMS OF THE INVERSES OF BINOMIAL COEFFICIENTS

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In this note, we discuss several sums of inverses of binomial coefficients. We evaluate these sums by application of a fundamental recurrence relation in much the same manner as sums of binomial coefficients may be treated. As an application, certain iterated integrals of the logarithm are evaluated.