In the case where the ring R is Z, the set of integers, we can determine the total number of different solutions mod (e, f), or $\frac{e}{f}$.

This number of solutions will be the smallest positive integer n such that

$$(nb_1e_1, 1) \equiv 0 \mod (e, f),$$

i.e., such that $e | nb_1e_1f$.

Now, as we can assume that e and f and a and b are mutually prime, this reduces to $i|n$, so the smallest n is i.

Thus in the ring of integers, the number of noncongruent solutions mod (e, f) of (1) is i.

Take, as an example,

$$15\tfrac{5}{39}x \equiv \tfrac{5}{6} \mod 20\tfrac{5}{52}.$$

Clearly, g.c.d. $\left(15\frac{5}{39}, 20\frac{5}{52}\right) = \frac{5}{156}\Big|\frac{5}{6}$, and we can obtain $x = -89$ as a solution to

$$4(15.39 + 5)x \equiv 26.5 \mod (60.52 + 15).$$

Now b_1 comes to 3 and e_1 to 209, so the simplest noncongruent positive integer solutions, mod $20\frac{5}{52}$, are 194, 821, 1448, 2075, and 2702.

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A RECURSION-TYPE FORMULA FOR SOME PARTITIONS

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If $p(n)$ denotes the number of unrestricted partitions of n, the following recurrence formula, known as Euler's identity, permits the computation of $p(n)$ if $p(k)$ is already known for $k < n$.

$$(1) \quad p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - - + + \cdots$$

$$= \sum_{j \neq 0} (-1)^{j+1} p\left(n - \frac{1}{2}(3j^2 + j)\right),$$

where the sum extends over all integers j, except $j = 0$, for which the arguments of the partition function are nonnegative.

Hickerson [1] gave a recursion-type formula for $q(n)$, the number of partitions of n into distinct parts, in terms of $p(k)$ for $k \leq n$, as follows,

$$(2) \qquad q(n) = \sum_{j=-\infty}^{\infty} (-1)^j p(n - (3j^2 + j)),$$

where the sum extends over all integers j for which the arguments of the partition function are nonnegative.

Alder and Muwafi [2] gave a recursion-type formula for $p'(0, k - r, 2k + a; n)$, the number of partitions of n into parts $\not\equiv 0, \pm(k - r) \mod 2k + a$, where $0 \leq r \leq k - 1$.

$$(3) \qquad p'(0,\, k - r,\, 2k + a;\, n) = \sum_{j=-\infty}^{\infty} (-1)^j p\left(n - \frac{(2k + a)j^2 + (2r + a)j}{2}\right)$$

where the sum extends over all integers j for which the arguments of the partition function are nonnegative. Letting $k = a = 1$ and $r = 0$, formula (3) reduces to Euler's identity; and letting $k = a = 2$ and $r = 0$, formula (3) reduces to Hickerson's formula (2).

Ewell [3] gave two recurrence formulas for $q(2\ell)$ and $q(2\ell + 1)$ for nonnegative integers ℓ in a slightly different, but equivalent, form to that in formula (2).

This paper presents a recursion-type formula for $p_k^*(n)$, the number of partitions of n into parts not divisible by k, where k is some given integer ≥ 1. It is shown that formulas (1) and (2) are special cases of formula (4) below.

Theorem: If $n \geq 0$, $k \geq 1$, and $p_k^*(n)$ is the number of partitions of n into parts not divisible by k, where $p_k^*(0) = 1$, then

$$(4) \qquad p_k^*(n) = \sum_{j=-\infty}^{\infty} (-1)^j p\left(n - \frac{k(3j^2 + j)}{2}\right),$$

where the sum extends over all integers j for which the arguments of the partition function are nonnegative.

Proof: The generating function for $p_k^*(n)$ is given by

$$\sum_{n=0}^{\infty} p_k^*(n)x^n = \frac{\prod_{j=1}^{\infty}(1 - x^{kj})}{\prod_{j=1}^{\infty}(1 - x^j)} = \sum_{r=0}^{\infty} p(r)x^r \prod_{j=1}^{\infty}(1 - x^{kj}).$$

By Euler's product formula, we have

$$\prod_{j=1}^{\infty}(1 - x^{kj}) = \sum_{j=-\infty}^{\infty}(-1)^j x^{\frac{kj(3j+1)}{2}}$$

Hence

$$\sum_{n=0}^{\infty} p_k^*(n)x^n = \sum_{r=0}^{\infty} p(r)x^r \sum_{j=-\infty}^{\infty}(-1)^j x^{\frac{kj(3j+1)}{2}}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{j=-\infty}^{\infty}(-1)^j p\left(n - \frac{kj(3j + 1)}{2}\right)\right\}x^n.$$

Equating coefficients on both sides of this equation, and noticing that $j = 0$ when $n = 0$, we get the required result in (4).

Corollary 1: If in Eq. (4) we let $k = 1$, then $p_1^*(n) = 0$, so that Eq. (4) becomes

$$0 = \sum_{j=-\infty}^{\infty}(-1)^j p\left(n - \frac{3j^2 + j}{2}\right),$$

from which Eq. (1) follows by moving the term corresponding to $j = 0$ to the left-hand side. Thus Eq. (1) becomes a special case of the theorem.

Corollary 2: If in Eq. (4) we let $k = 2$, then $p_2^*(n)$ denotes the number of partitions of n into parts not divisible by 2, and hence it is equal to the number of partitions of n into odd or distinct parts. Thus $p_2^*(k) = q(n)$, and Eq. (4) reduces to (2). Hence Eq. (2) is a special case of the theorem.

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PRIMITIVE PYTHAGOREAN TRIPLES AND THE INFINITUDE OF PRIMES

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A *primitive Pythagorean triple* is a triple of natural numbers (x, y, z) such that $x^2 + y^2 = z^2$ and $(x, y) = 1$. It is well known [1, pp. 4-6] that all primitive Pythagorean triples are given, without duplication, by

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2,$$

where m and n are relatively prime natural numbers which are of opposite parity and satisfy $m > n$. Conversely, if m and n are relatively prime natural numbers which are of opposite parity and $m > n$, then the above formulas yield a primitive Pythagorean triple. In this note I will refer to m and n as the generators of the triple (x, y, z) and I will refer to x and y as the legs of the triple.

A study of the sums of the legs of primitive Pythagorean triples leads to the following interesting variation of Euclid's famous proof that there are infinitely many primes.

Suppose there is a largest prime, say p_k. Let m be the product of this finite list of primes and let $n = 1$. Then $(m, n) = 1$, $m > n$, and they are of opposite parity. Thus m and n generate a primitive Pythagorean triple according to the above formulas. If $x + y$ is prime, it follows from

$$x + y = 2mn + m^2 - n^2 = 2(2 \cdot 3 \cdot \cdots \cdot p_k) + (2 \cdot 3 \cdot \cdots \cdot p_k)^2 - 1 > p_k^2$$

that $x + y$ is a prime greater than p_k. If $x + y$ is composite, it must have a prime divisor greater than p_k. This last statement follows from the fact that every prime $q \le p_k$ divides m and hence divides x. If q divides $x + y$, then it divides y, which contradicts the fact that (x, y, z) is a primitive Pythagorean triple. Thus the assumption that p_k is the largest prime is false.

By noting that

$$\begin{aligned} y - x &= (2 \cdot 3 \cdot \cdots \cdot p_k)^2 - 1 - 2(2 \cdot 3 \cdot \cdots \cdot p_k) \\ &= 2(2 \cdot 3 \cdot \cdots \cdot p_k)(3 \cdot \cdots \cdot p_k - 1) - 1 > p_k, \end{aligned}$$

a similar proof can be constructed by using the difference of the legs of the primitive Pythagorean triple (x, y, z).

The following lemma will be useful in proving that there are infinitely many primes of the form $8t \pm 1$.

Lemma: If (x, y, z) is a primitive Pythagorean triple and p is a prime divisor of $x + y$ or $|x - y|$, then p is of the form $8t \pm 1$.

Proof: Suppose p divides $x + y$ or $|x - y|$. Note that this implies

$$(x, p) = (y, p) = 1, \quad \text{and} \quad x \equiv \pm y \pmod{p}$$

so that