If we fix i, we may write $m = hp_i^{\alpha_i}$ with $h > 1$ and $(h, p_i) = 1$. The question will then be settled if we show that there are *no* carries when adding $p_i^{\alpha_i}$ to $m - p_i^{\alpha_i} = (h - 1)p_i^{\alpha_i}$ in base p. Since the only nonzero digit in the representation of $p_i^{\alpha_i}$ to base $p_i^{\alpha_i}$ is the 1 that multiplies p_i^{α}, we need consider only the digit that multiplies $p_i^{\alpha_i}$ in the base p_i representation of $(h - 1)p_i^{\alpha_i}$. Indeed, it is clear that we have a carry if and only if $h - 1 = qp_i + (p_i - 1)$ for some integer q. But this is so if and only if $h = (q + 1)p_i$, and this contradicts the fact that $(h, p_i) = 1$. Thus,

$$p_i \nmid \binom{m}{p_i^{\alpha_i}}$$

for $1 \leq i \leq r$, and the proof is complete.

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ON THE NUMBER OF FIBONACCI PARTITIONS OF A SET

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1. PARTITIONS OF $\overline{n}$ IN FIBONACCI SETS

Let $\overline{n}: = \{1, 2, \ldots, n\}$. It is well known [1] that the number of sets $A \subseteq \overline{n}$, with

(1) $$i, j \in A,\ i \neq j \text{ implies } |i - j| \geq 2,$$

is the Fibonacci number F_{n+1}. ($F_0 = F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$.)

A set $A \subseteq \overline{n}$ with the property (1) will be called a Fibonacci set.

A partition of $\overline{n}$ is a family of disjoint (nonempty) subsets of $\overline{n}$ whose union is $\overline{n}$. The number of partitions of $\overline{n}$ is B_n, the nth Bell number [2].

In this section the number C_n of partitions of $\overline{n}$ in Fibonacci subsets will be considered. There exists an interesting connection with B_n.

Theorem 1: $C_n = B_{n-1}$.

Proof: This will be proved by arguments analogous to Rota's in [2]. First, the number of functions $f : \overline{n} \to U$ (U has u elements) with $f(i) \neq f(i + 1)$ for all i is determined: for $f(1)$ there are u possibilities; for $f(2)$ there are $u - 1$ possibilities; for $f(3)$ there are $u - 1$ possibilities, and so on. The desired number of functions is $u(u - 1)^{n-1}$.

These functions are partitioned with respect to their kernels. (Note that exactly those kernels appear which are Fibonacci sets!)

(2) $$\sum (u)_{N(\pi)} = u(n - 1)^{n-1},$$

the sum is extended over all kernels π, and $N(\pi)$ denotes the number of distinct subsets of π.

Now let L be the functional defined by $(u)_n \to 1$ for all n. Then, from (2),

$$L\left(\sum (u)_{N(\pi)}\right) = C_n = L(u(u-1)^{n-1}). \tag{3}$$

In [2] it is proved that $L(u \cdot p(u-1)) = L(p(u))$ holds for all polynomials p. With $p(u) = u^{n-1}$,

$$C_n = L(u(u-1)^{n-1}) = L(u^{n-1}) = B_{n-1}.$$

(The last equality is the essential result of [2].)

At this time it is legitimate to ask of a natural bijection φ from the partitions of $\overline{n}$ to the Fibonacci partitions of $\overline{n+1}$. φ and φ^{-1} are given by the following algorithms (due to F. J. Urbanek).

Algorithm for φ:

A1. $n + 1$ is adjoined to the given partition in a new class.
A2. Do Step A3 for all classes except the one of $n + 1$.
A3. Run through the class in decreasing order. If with the considered number i, $i + 1$ is also in the same class, give i in the class of $n + 1$.

Example: $1\ 2\ 3\ 5|4\ 6\ 7|8\ 9 \to 1\ 2\ 3\ 5|4\ 6\ 7|8\ 9|10 \to 1\ 3\ 5|4\ 6\ 7|8\ 9|2\ 10$
$\to 1\ 3\ 5|4\ 7|8\ 9|2\ 6\ 10 \to 1\ 3\ 5|4\ 7|9|2\ 6\ 8\ 10.$

Algorithm for φ^{-1}: The number $n + 1$ is erased; the other numbers in this class are to be distributed: If $i + 1$ has its place and i is to be distributed, give i in the class of $i + 1$.

Example: $138|24|6|579 \to 1378|24|56.$

It is not difficult to see that φ and φ^{-1} are inverse and that only φ^{-1} preserves the partial order of partitions (with respect to refinement).

2. A GENERALIZATION: d-FIBONACCI SETS

A d-Fibonacci set $A \subseteq \overline{n}$ has the property

$$i,\ j \in A,\ i \neq j \text{ implies } |i - j| \geq d. \tag{4}$$

Let $C_n^{(d)}$ be the number of d-Fibonacci partitions. ($C_n^{(2)} = C_n$, $C_n^{(1)} = B_n$.)

Theorem 2: $C_n^{(d)} = B_{n+1-d}$.

Proof: First the number of functions $f: \overline{n} \to U$ with

$$|\{f(i), f(i+1), \ldots, f(i+d-1)\}| = d \text{ for all } i$$

is considered. By the same argument as in Section 1, this number is

$$(u)_{d-1}(u - d + 1)^{n+1-d}.$$

Again

$$\sum (u)_{N(\pi)} = (u)_{d-1}(u - d + 1)^{n+1-d}, \tag{5}$$

where the summation ranges over all d-Fibonacci partitions of $\overline{n}$. Applying the functional L on (5) yields

$$C_n^{(d} = L((u)_{d-1}(u - k + 1)^{n+1-d}). \tag{6}$$

As in [2],

$$L((u)_{d-1}p(u - d + 1)) = L(p(u)) \tag{7}$$

holds for all polynomials p. With $p(u) = u^{n+1-d}$ it follows from (6) and (7) that

$$C_n^{(d)} = L\big((u)_{d-1}(u - d + 1)^{n+1-d}\big) = L(u^{n+1-d}) = B_{n+1-d}.$$

It is possible to construct a bijection φ from the partitions of $\overline{n}$ to the d-Fibonacci partitions of $\overline{n + d - 1}$ in a way similar to that given in the previous section; however, this is more complicated to describe and therefore is omitted.

3. A GENERALIZATION OF THE FIBONACCI NUMBERS

The fact that F_{n+1} is the number of Fibonacci subsets of $\overline{n}$ can be seen as the starting point to define the numbers $F_n^{(s)}$ $(s \in N)$:

$F_{n+1}^{(s)}$ is defined to be the number of $(A_1, \ldots, A_s)$ with $A_i \subseteq \overline{n}$ and $A_i \cap A_j \neq \emptyset$ for $i \neq j$. The recurrence

$$F_{n+1}^{(s)} = sF_n^{(s)} + F_{n-1}^{(s)},\ F_1^{(s)} = 1,\ F_2^{(s)} = 1 + s$$

can be established as follows:

First, $F_{n+1}^{(s)}$ can be expressed as the number of functions

$$f : \overline{n} \to \{\varepsilon, a_1, \ldots, a_s\}$$

with $f(i) = f(i + 1) = a_j$ is impossible. If $f(n) = \varepsilon$, the contribution to $F_{n+1}^{(s)}$ is $F_n^{(s)}$. If $f(n) = a_i$, the contribution is $F_n^{(s)}$ minus the number of functions

$$f: \overline{n - 1} \to \{\varepsilon, a\ , \ldots, a_s\}$$

with $f(n - 1) = a_i$. Taken all together,

$$F_{n+1}^{(s)} = F_n^{(s)} + s[F_n^{(s)} - F_{n-1}^{(s)} + F_{n-2}^{(s)} - + \cdots]. \tag{8}$$

Also

$$F_{n+2}^{(s)} = F_{n+1}^{(s)} + s[F_{n+1}^{(s)} - F_n^{(s)} + F_{n-1}^{(s)} - + \cdots]. \tag{9}$$

Adding (8) and (9) gives the result. An explicit expression is

$$F_n^{(s)} = \frac{1}{\sqrt{s^2 + 4}}\left[\left(\frac{s + \sqrt{s^2 + 4}}{2}\right)^{n+1} - \left(\frac{s - \sqrt{s^2 + 4}}{2}\right)^{n+1}\right].$$

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(continued from page 406)

Added in proof. Other explicit formulas for $P(n, s)$ were obtained in the paper "Enumeration of Permutations by Sequences," *The Fibonacci Quarterly* 16 (1978): 259-68. See also L. Comtet, *Advanced Combinatorics* (Dordrecht & Boston: Reidel, 1974), pp. 260-61.

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