REFERENCES

1. Dean R. Hickerson. "Recursion-Type for Partitions into Distinct Parts." *The Fibonacci Quarterly* 11 (1973):307-12.
2. Henry L. Alder & Amin A. Muwafi. "Generalizations of Euler's Recurrence Formula for Partitions." *The Fibonacci Quarterly* 13 (1975):337-39.
3. John A. Ewell. "Partition Recurrences." *J. Combinatorial Theory, Series A.* 14 (1973):125-27.

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PRIMITIVE PYTHAGOREAN TRIPLES AND THE INFINITUDE OF PRIMES

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A *primitive Pythagorean triple* is a triple of natural numbers (x, y, z) such that $x^2 + y^2 = z^2$ and $(x, y) = 1$. It is well known [1, pp. 4-6] that all primitive Pythagorean triples are given, without duplication, by

$$x = 2mn, \; y = m^2 - n^2, \; z = m^2 + n^2,$$

where m and n are relatively prime natural numbers which are of opposite parity and satisfy $m > n$. Conversely, if m and n are relatively prime natural numbers which are of opposite parity and $m > n$, then the above formulas yield a primitive Pythagorean triple. In this note I will refer to m and n as the generators of the triple (x, y, z) and I will refer to x and y as the legs of the triple.

A study of the sums of the legs of primitive Pythagorean triples leads to the following interesting variation of Euclid's famous proof that there are infinitely many primes.

Suppose there is a largest prime, say p_k. Let m be the product of this finite list of primes and let $n = 1$. Then $(m, n) = 1$, $m > n$, and they are of opposite parity. Thus m and n generate a primitive Pythagorean triple according to the above formulas. If $x + y$ is prime, it follows from

$$x + y = 2mn + m^2 - n^2 = 2(2 \cdot 3 \cdot \cdots \cdot p_k) + (2 \cdot 3 \cdot \cdots \cdot p_k)^2 - 1 > p_k^2$$

that $x + y$ is a prime greater than p_k. If $x + y$ is composite, it must have a prime divisor greater than p_k. This last statement follows from the fact that every prime $q \le p_k$ divides m and hence divides x. If q divides $x + y$, then it divides y, which contradicts the fact that (x, y, z) is a primitive Pythagorean triple. Thus the assumption that p_k is the largest prime is false.

By noting that

$$\begin{aligned} y - x &= (2 \cdot 3 \cdot \cdots \cdot p_k)^2 - 1 - 2(2 \cdot 3 \cdot \cdots \cdot p_k) \\ &= 2(2 \cdot 3 \cdot \cdots \cdot p_k)(3 \cdot \cdots \cdot p_k - 1) - 1 > p_k, \end{aligned}$$

a similar proof can be constructed by using the difference of the legs of the primitive Pythagorean triple (x, y, z).

The following lemma will be useful in proving that there are infinitely many primes of the form $8t \pm 1$.

Lemma: If (x, y, z) is a primitive Pythagorean triple and p is a prime divisor of $x + y$ or $|x - y|$, then p is of the form $8t \pm 1$.

Proof: Suppose p divides $x + y$ or $|x - y|$. Note that this implies

$$(x, p) = (y, p) = 1, \quad \text{and} \quad x \equiv \pm y \pmod{p}$$

so that

$$2x^2 \equiv x^2 + y^2 \equiv z^2 \pmod{p}.$$

By definition, x^2 is a quadratic residue of p. The above congruence implies $2x^2$ is also a quadratic residue of p. If p were of the form $8t \pm 3$, then 2 would be a quadratic nonresidue of p and since x^2 is a quadratic residue of p, $2x^2$ would be a quadratic nonresidue of p, a contradiction. Thus p must be of the form $8t \pm 1$.

Now, if we assume that there is a finite number of primes of the form $8t \pm 1$, and if we let m be the product of these primes, then we obtain a contradiction by imitating the above proof that there are infinitely many primes.

REFERENCE

1. W. Sierpinski. "Pythagorean Triangles." *Scripta Mathematica Studies*, No. 9. New York: Yeshiva University, 1964.

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AN APPLICATION OF PELL'S EQUATION

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The following problem solution is a good classroom presentation or exercise following a discussion of Pell's equation.

Statement of the Problem

Find all natural numbers a and b such that

$$\frac{a(a+1)}{2} = b^2.$$

An alternate statement of the problem is to ask for all triangular numbers which are squares.

Solution of the Problem

$$\frac{a(a+1)}{2} = b^2 \iff a^2 + a = 2b^2 \iff a^2 + a - 2b^2 = 0 \iff a = \frac{-1 \pm \sqrt{1+8b^2}}{2} \iff \exists$$

an odd integer t such that $t^2 - 2(2b)^2 = 1$.

This is Pell's equation with fundamental solution [1, p. 197] $t = 3$ and $2b = 2$ or, equivalently, $t = 3$ and $b = 1$. Note that $t = 3$ implies

$$a = \frac{-1 \pm 3}{2},$$

but, according to the following theorem, we may discard $a = -2$. Also note that t is odd.

Theorem 1: If D is a natural number that is not a perfect square, the Diophantine equation $x^2 - Dy^2 = 1$ has infinitely many solutions x, y.

All solutions with positive x and y are obtained by the formula

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n,$$

where x_1, y_1 is the fundamental solution of $x^2 - Dy^2 = 1$ and where n runs through all natural numbers.