A GENERALIZED EXTENSION OF SOME FIBONACCI-LUCAS IDENTITIES TO PRIMITIVE UNIT IDENTITIES

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This paper originated from an attempt to extend many of the elementary Fibonacci-Lucas identities, whose subscripts had a common odd or even difference to, first, other Type I real quadratic fields and, then, to the other three types of real quadratic field fundamental units. For example, the Edouard Lucas identity $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$ becomes, in the Type I real quadratic field,

$$\left(\sqrt{61};\ \alpha = \frac{39 + 5\sqrt{61}}{2}\right)F_{n+1}^3 + 39F_n^3 - F_{n-1}^3 = (5)(195)F_{3n}.$$

This suggests the Type I extension identity $F_{n+1}^3 + L_1F_n^3 - F_{n-1}^3 = F_1F_2F_{3n}$ and the Type I generalization: $F_{n+2r+1}^3 + L_{2r+1}F_n^3 - F_{n-2r-1}^3 = F_{2r+1}F_{4r+2}F_{3n}$. The Ezekiel Ginsburg identity $F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n}$ becomes, in the Type I real quadratic field,

$$(\sqrt{61})F_{n+2}^3 - 1523F_n^3 + F_{n-2}^3 = (195)(296985)F_{3n}.$$

This suggests the Type I identity extension $F_{n+2}^3 - L_2F_n^3 + F_{n-2}^3 = F_2F_4F_{3n}$ and the Type I generalization: $F_{n+2r}^3 - L_{2r}F_n^3 + F_{n-2r}^3 = F_{2r}F_{4r}F_{3n}$.

The transformation from these Type I identities to Type III identities can be represented as

$$\text{(I) } F_n \leftrightarrow \text{(III) } 2F_n \quad \text{or} \quad \text{(I) } L_n \leftrightarrow \text{(III) } 2L_n.$$

The transformation from Type I to Type II and Type III to Type IV for identities in which there is a common even subscript difference $2r$ can be represented as

$$\text{(I, III) } F_{2r} \leftrightarrow \text{(II, IV) } F_r,\ L_{2r} \leftrightarrow L_r,\ F_{n+2r} \leftrightarrow F_{n+r},\ \text{and } L_{n+2r} \leftrightarrow L_{n+r}.$$

I. *Type I* primitive units are given by

$$\alpha = \frac{a + b\sqrt{D}}{2},\ \beta = \frac{a - b\sqrt{D}}{2},\ \alpha\beta = -1,\ D \equiv 5 \text{ (modulo 8)},$$

$$a^2 - b^2D = -4,\ a \text{ and } b \text{ are odd.}$$

$$\left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},\ F_n = \frac{1}{\sqrt{D}}(\alpha^n - \beta^n),\ L_n = \alpha^n + \beta^n.$$

F_n and L_n are also given by the finite difference sequences:

$$F_{n+2} = aF_{n+1} + F_n,\ F_1 = b,\ F_2 = ab;$$

$$L_{n+2} = aL_{n+1} + L_n,\ L_1 = a,\ L_2 = a^2 + 2.$$

II. *Type II* primitive units are given by

$$\alpha = \frac{a + b\sqrt{D}}{2},\ \beta = \frac{a - b\sqrt{D}}{2},\ \alpha\beta = 1,\ D \equiv 5 \text{ (modulo 8)},$$

$$a^2 - b^2D = 4,\ a^2 - b^2D \neq -4,\ a \text{ and } b \text{ are odd.}$$

$$\left(\frac{a + b\sqrt{D}}{2}\right)^n = \frac{L_n + F_n\sqrt{D}}{2},\ F_n = \frac{1}{\sqrt{D}}(\alpha^n - \beta^n),\ L_n = \alpha^n + \beta^n.$$

F_n and L_n are also given by the finite difference sequences:

$$F_{n+2} = aF_{n+1} - F_n,\ F_1 = b,\ F_2 = ab;$$
$$L_{n+2} = aL_{n+1} - L_n,\ L_1 = a,\ L_2 = a^2 - 2.$$

III. *Type III* primitive units are given by

$$\alpha = a + b\sqrt{D},\ \beta = a - b\sqrt{D},\ \alpha\beta = -1,\ a^2 - b\sqrt{D} = -1.$$

$$(a + b\sqrt{D})^n = L_n + F_n\sqrt{D},\ F_n = \frac{1}{2\sqrt{D}}(\alpha^n - \beta^n),\ L_n = \frac{1}{2}(\alpha^n + \beta^n).$$

F_n and L_n are also given by the finite difference sequences:

$$F_{n+2} = 2aF_{n+1} + F_n,\ F_1 = b,\ F_2 = 2ab;$$
$$L_{n+2} = 2aL_{n+1} + L_n,\ L_1 = a,\ L_2 = 2a^2 + 1.$$

IV. *Type IV* primitive units are given by

$$\alpha = a + b\sqrt{D},\ \beta = a - b\sqrt{D},\ \alpha\beta = 1,\ a^2 - b^2D = 1,\ a^2 - b^2D \neq -1.$$

$$(a + b\sqrt{D})^n = L_n + F_n\sqrt{D},\ F_n = \frac{1}{2\sqrt{D}}(\alpha^n - \beta^n),\ L_n = \frac{1}{2}(\alpha^n + \beta^n).$$

F_n and L_n are also given by the finite difference sequences:

$$F_{n+2} = 2aF_{n+1} - F_n,\ F_1 = b,\ F_2 = 2ab;$$
$$L_{n+2} = 2aL_{n+1} - L_n,\ L_1 = a,\ L_2 = 2a^2 - 1.$$

1. (a) Fibonacci-Lucas identity used: $F_n + L_n = 2F_{n+1}$
   (b) Type I extension: $aF_n + bL_n = 2F_{n+1}$
   (c) Generalizations:

<u>Types I & II</u> $L_mF_n + F_mL_n = 2F_{m+n}$

<u>Types III & IV</u> $L_mF_n + F_mL_n = F_{m+n}$

2. (a) Fibonacci-Lucas identity used: $L_n - F_n = 2F_{n-1}$
   (b) Type I extension: $bL_n - aF_n = 2F_{n-1}$
   (c) Generalizations:

<u>Type I</u> $F_mL_n - L_mF_n = 2(-1)^{m+1}F_{n-m}$

<u>Type II</u> $F_nL_m - F_mL_n = 2F_{n-m}$

<u>Type III</u> $F_mL_n - L_mF_n = (-1)^{m+1}F_{n-m}$

<u>Type IV</u> $F_nL_m - F_mL_n = F_{n-m}$

3. (a) Fibonacci-Lucas identity used: $F_{n+3}^2 + F_n^2 = 2(F_{n+2}^2 + F_{n+1}^2)$
   (b) Type I extension: $b(F_{n+3}^2 + F_n^2) = F_3(F_{n+2}^2 + F_{n+1}^2)$
   (c) Generalizations:

<u>Types I & III</u> $F_{2r-1}(F_{n+4m-1}^2 + F_n^2) = F_{4m-1}(F_{n+2m+r-1}^2 + F_{n+2m-r}^2)$

$F_{2r-1}(L_{n+4m-1}^2 + L_n^2) = F_{4m-1}(L_{n+2m+r-1}^2 + L_{n+2m-r}^2)$

<u>Types II & IV</u> $F_{2r-1}(F_{n+4m-1}^2 - F_n^2) = F_{4m-1}(F_{n+2m+r-1}^2 - F_{n+2m-r}^2)$

$L_{2r-1}(L_{n+4m-1}^2 - L_n^2) = F_{4m-1}(L_{n+2m+r-1}^2 - L_{n+2m-r}^2)$

4. (a) Fibonacci-Lucas identity used:

$$F_{n+3}F_{n+4} + F_nF_{n+1} = 2(F_{n+2}F_{n+3} + F_{n+1}F_{n+2})$$

(b) Type I extension:

$$b(F_{n+3}F_{n+4} + F_nF_{n+1}) = F_3(F_{n+2}F_{n+3} + F_{n+1}F_{n+2})$$

(c) Generalizations:

Types I & III

$$F_{2r-1}(F_{n+4m-1}F_{n+4m} + F_nF_{n+1}) = F_{4m-1}(F_{n+2m+r-1}F_{n+2m+r} + F_{n+2m-r}F_{n+2m-r+1})$$
$$F_{2r-1}(L_{n+4m-1}L_{n+4m} + L_nL_{n+1}) = F_{4m-1}(L_{n+2m+r-1}L_{n+2m+r} + L_{n+2m-r}L_{n+2m-r+1})$$

Types II & IV

$$F_{2r-1}(F_{n+4m-1}F_{n+4m} - F_nF_{n+1}) = F_{4m-1}(F_{n+2m+r-1}F_{n+2m+r} - F_{n+2m-r}F_{n+2m-r+1})$$
$$F_{2r-1}(L_{n+4m-1}L_{n+4m} - L_nL_{n+1}) = F_{4m-1}(L_{n+2m+r-1}L_{n+2m+r} - L_{n+2m-r}L_{n+2m-r+1})$$

5. (a) Fibonacci-Lucas identity used: $F_{2m} + F_m^2 = 2F_mF_{m+1}$
(b) Type I extension: $bF_{2m} + aF_{2m}^2 = 2F_mF_{m+1}$
(c) Generalizations:

Type I

$$F_rF_{2m} + L_rF_m^2 = 2F_mF_{m+r}$$
$$DF_rF_{2m} + L_rL_m^2 = 2L_mL_{m+r}$$

Type II

$$F_rF_{2m} + L_rF_m^2 = 2F_mF_{m+r}$$
$$DF_rF_{2m} + L_rL_m^2 = 2L_mL_{m+r}$$

Type III

$$F_rF_{2m} + 2L_rF_m^2 = 2F_mF_{m+r}$$
$$DF_rF_{2m} + 2L_rL_m^2 = 2L_mL_{m+r}$$

Type IV

$$F_rF_{2m} + 2L_rF_m^2 = 2F_mF_{m+r}$$
$$DF_rF_{2m} + 2L_rL_m^2 = 2L_mL_{m+r}$$

6. (a) Fibonacci-Lucas identity used: $F_{2m} - F_m^2 = 2F_mF_{m-1}$
(b) Type I extension: $bF_{2m} - aF_m^2 = 2F_mF_{m-1}$
(c) Generalizations:

Type I

$$F_rF_{2m} - L_rF_m^2 = 2(-1)^{r+1}F_mF_{m-r}$$
$$DF_rF_{2m} - L_rL_m^2 = 2(-1)^{r+1}L_mL_{m-r}$$

Type II

$$F_rF_{2m} - L_rL_m^2 = -2L_mL_{m-r}$$
$$DF_rF_{2m} - L_rL_m^2 = -2L_mL_{m-r}$$

Type III

$$F_rF_{2m} - 2L_rF_m^2 = 2(-1)^{r+1}F_mF_{m-r}$$
$$DF_rF_{2m} - 2L_rL_m^2 = 2(-1)^{r+1}L_mL_{m-r}$$

Type IV

$$F_rF_{2m} - 2L_rF_m^2 = -2F_mF_{m-r}$$
$$DF_rF_{2m} - 2L_rL_m^2 = -2L_mL_{m-r}$$

7. (a) Fibonacci-Lucas identity used: $L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}$
(b) Type I extension: $b^2L_n^2 - a^2F_n^2 = 4F_{n-1}F_{n+1}$

(c) Generalizations:

Types I & III

$$F_r^2L_n^2 - L_r^2F_n^2 = 4(-1)^{r+1}F_{n+r}F_{n-r}, \text{ I}; \; (-1)^{r+1}F_{n+r}F_{n-r}, \text{ III}$$
$$D^2F_r^2F_n^2 - L_r^2L_n^2 = 4(-1)^{r+1}L_{n+r}L_{n-r}, \text{ I}; \; (-1)^{r+1}L_{n+r}L_{n-r}, \text{ III}$$

Types II & IV

$$F_r^2F_n^2 - L_r^2F_n^2 = -4F_{n+r}F_{n-r}, \text{ II}; \; -F_{n+r}F_{n-r}, \text{ IV}$$
$$D^2F_r^2F_n^2 - L_r^2L_n^2 = -4L_{n+r}L_{n-r}, \text{ II}; \; -L_{n+r}L_{n-r}, \text{ IV}$$

8. (a) Fibonacci-Lucas identity used: $L_{2n}L_{2n+2} - 5F_{2n+1}^2 = 1$
(b) Type I extension: $L_{2n}L_{2n+2} - DF_{2n+1}^2 = a^2$
(c) Generalizations:

All Types

$$L_{2n}L_{2n+2r} - DF_{2n+r}^2 = L_r^2$$
$$L_{2n+r}^2 - DF_{2n}F_{2n+2r} = L_r^2$$

9. (a) Fibonacci-Lucas identity used:

$$F_{r+m+n} = F_{m+1}F_{n+1}F_{r+1} + F_mF_nF_r - F_{m-1}F_{n-1}F_{r-1}$$

(b) Type I extension:

$$ab^2F_{r+m+n} = F_{m+1}F_{n+1}F_{r+1} + aF_mF_nF_r - F_{m-1}F_{n-1}F_{r-1}$$

(c) Generalizations:

Type I

$$F_{m+2t+1}F_{n+2t+1}F_{r+2t+1} + L_{2t+1}F_mF_nF_r - F_{m-2t-1}F_{n-2t-1}F_{r-2t-1}$$
$$= \frac{1}{D}(L_{6t+3} + L_{2t+1})F_{m+n+r} = F_{2t+1}F_{4t+2}F_{m+n+r}$$

$$L_{m+2t+1}L_{n+2t+1}L_{r+2t+1} + L_{2t+1}L_mL_nL_r - L_{m-2t-1}L_{n-2t-1}L_{r-2t-1}$$
$$= (L_{6t+3} + L_{2t+1})L_{m+n+r} = DF_{2t+1}F_{4t+2}F_{m+n+r}$$

$$F_{m+2t}F_{n+2t}F_{r+2t} - L_{2t}F_mF_nF_r + F_{m-2t}F_{n-2t}F_{r-2t} = \frac{1}{D}(L_{6t} - L_{2t})F_{m+n+r}$$

$$L_{m+2t}L_{n+2t}L_{r+2t} - L_{2t}L_mL_nL_r + L_{m-2t}L_{n-2t}L_{r-2t} = (L_{6t} - L_{2t})L_{m+n+r}$$
$$= DF_{2t}F_{4t}L_{m+n+r}$$

Type II

$$F_{m+t}F_{n+t}F_{r+t} - L_tF_mF_nF_r + F_{m-t}F_{n-t}F_{r-t} = \frac{1}{D}(L_{3t} - L_t)F_{m+n+r}$$
$$L_{m+t}L_{n+t}L_{r+t} - L_tL_mL_nL_r + L_{m-t}L_{n-t}L_{r-t} = (L_{3t} - L_t)L_{m+n+r}$$

Type III

$$F_{m+2t+1}F_{n+2t+1}F_{r+2t+1} + 2L_{2t+1}F_mF_nF_r - F_{m-2t-1}F_{n-2t-1}F_{r-2t-1}$$
$$= \frac{1}{2D}(L_{6t+3} + L_{2t+1})F_{m+n+r} = F_{2t+1}F_{4t+2}F_{m+n+r}$$

$$L_{m+2t+1}L_{n+2t+1}F_{r+2t+1} + 2L_{2t+1}L_mL_nL_r - L_{m-2t-1}L_{n-2t-1}L_{r-2t-1}$$
$$= \frac{1}{2}(L_{6t+3} + L_{2t+1})L_{m+n+r} = DF_{2t+1}F_{4t+2}L_{m+n+r}$$

$$F_{m+2t}F_{n+2t}F_{r+2t} - 2L_{2t}F_mF_nF_r + F_{m-2t}F_{n-2t}F_{r-2t} = \frac{1}{2D}(L_{6t} - L_{2t})F_{m+n+r}$$
$$= F_{2t}F_{4t}F_{m+n+r}$$

$$L_{m+2t}L_{n+2t}L_{r+2t} - 2L_{2t}L_mL_nL_r + L_{m-2t}L_{n-2t}L_{r-2t} = \frac{1}{2}(L_{6t} - L_{2t})F_{m+n+r}$$

Type IV $F_{m+t}F_{n+t}F_{r+t} - 2L_tF_mF_nF_r + F_{m-t}F_{n-t}F_{r-t} = \frac{1}{2D}(L_{3t} - L_t)F_{m+n+r}$

$L_{m+t}L_{n+t}L_{r+t} - 2L_tL_mL_nL_r + L_{m-t}L_{n-t}L_{r-t} = \frac{1}{2}(L_{3t} - L_t)L_{m+n+r}$

10. (a) Fibonacci-Lucas identity used:

$$F^2_{n+1} + F^2_n + F^2_{n-1} = 2(F^2_{n+1} - F_nF_{n-1})$$

(b) Type I extension:

$$F^2_{n+1} + a^2F^2_n + F^2_{n-1} = 2(F^2_{n+1} - aF_nF_{n-1})$$

(c) Generalizations:

Type I $F^2_{n+2r+1} + L^2_{2r+1}F^2_n + F^2_{n-2r-1} = 2(F^2_{n+2r+1} - L_{2r+1}F_{n-2r-1}F_n)$

$L^2_{n+2r+1} + L^2_{2r+1}L^2_n + L^2_{n-2r-1} = 2(L^2_{n+2r+1} - L_{2r+1}L_{n-2r-1}L_n)$

$F^2_{n+2r} + L^2_{2r}F^2_n + F^2_{n-2r} = 2(F^2_{n+2r} + L_{2r}F_{n-2r}F_n)$

$L^2_{n+2r} + L^2_{2r}L^2_n + L^2_{n-2r} = 2(L^2_{n+2r} + L_{2r}L_{n-2r}L_n)$

Type II $F^2_{n+r} + L^2_rF^2_n + F^2_{n-r} = 2(F^2_{n+r} + L_rF_nF_{n-r})$

$L^2_{n+r} + L^2_rL^2_n + L^2_{n-r} = 2(L^2_{n+r} + L_rL_nL_{n-r})$

Type III $F^2_{n+2r+1} + 4L^2_{2r+1}F^2_n + F^2_{n-2r-1} = 2(F^2_{n+2r+1} - 2L_{2r+1}F_{n-2r-1}F_n)$

$L^2_{n+2r+1} + 4L^2_{2r+1}L^2_n + L^2_{n-2r-1} = 2(L^2_{n+2r+1} - 2L_{2r+1}L_{n-2r-1}L_n)$

$F^2_{n+2r} + 4L^2_{2r}F^2_n + F^2_{n-2r} = 2(F^2_{n+2r} + 2L_{2r}F_{n-2r}F_n)$

$L^2_{n+2r} + 4L^2_{2r}L^2_n + L^2_{n-2r} = 2(L^2_{n+2r} + 2L_{2r}L_{n-2r}L_n)$

Type IV $F^2_{n+r} + 4L^2_rF^2_n + F^2_{n-r} = 2(F^2_{n+r} + 2L_rF_nF_{n-r})$

$L^2_{n+r} + 4L^2_rL^2_n + L^2_{n-r} = 2(L^2_{n+r} + 2L_rL_nL_{n-r})$

11. (a) Fibonacci-Lucas identity used:

$$F^3_{n+2} = F^3_n + F^3_{n+1} + 3F_nF_{n+1}F_{n+2}$$

(b) Type I extension:

$$F^3_{n+2} = F^3_n + a^3F^3_{n+1} + 3aF_nF_{n+1}F_{n+2}$$

(c) Generalizations:

Type I $F^3_{n+2r+1} = F^3_{n-2r-1} + L^3_{2r+1}F^3_n + 3L_{2r+1}F\ F_{n+2r+1}F_{n-2r-1}$

$L^3_{n+2r+1} = L^3_{n-2r-1} + L^3_{2r+1}L^3_n + 3L_{2r+1}L\ L_{n+2r+1}L_{n-2r-1}$

$F^3_{n+2t} = L^3_{2t}F^3_n - F^3_{n-2t} - 3L_{2t}F_{n-2t}F_nF_{n+2t}$

$L^3_{n+2t} = L^3_{2t}L^3_n - L^3_{n-2t} - 3L_{2t}L_{n-2t}L_nL_{n+2t}$

Type II $F^3_{n+r} = L^3_rF^3_n - F^3_{n-r} - 3L_rF_nF_{n-r}F_{n+r}$

$L^3_{n+r} = L^3_rL^3_n - L^3_{n-r} - 3L_rL_nL_{n-r}L_{n+r}$

Type III $F^3_{n+2r+1} = F^3_{n-2r-1} + 8L^3_{2r+1}F^3_n + 6L_{2r+1}F_nF_{n+2r+1}F_{n-2r-1}$

$L^3_{n+2r+1} = L^3_{n-2r-1} + 8L^3_{2r+1}L^3_n + 6L_{2r+1}L_nL_{n+2r+1}L_{n-2r-1}$

$F^3_{n+2t} = 8L^3_{2t}F^3_n - F^3_{n-2t} - 6L_{2t}F_{n-2t}F_nF_{n+2t}$

$L^3_{n+2t} = 8L^3_{2t}L^3_n - L^3_{n-2t} - 6L_{2t}L_{n-2t}L_nL_{n+2t}$

<u>Type IV</u>

$$F^3_{n+r} = 8L^3_rF^3_n - F^3_{n-r} - 6L_rF_nF_{n-r}F_{n+r}$$
$$L^3_{n+r} = 8L^3_rL^3_n - L^3_{n-r} - 6L_rL_nL_{n-r}L_{n+r}$$

12. (a) Fibonacci-Lucas identity used:

$$F^4_{n+1} + F^4_n + F^4_{n-1} = 2[F^2_{n+1} - F_nF_{n-1}]^2$$

(b) Type I extension:

$$F^4_{n+1} + a^4F^4_n + F^4_{n-1} = 2[F^2_{n+1} - aF_nF_{n-1}]^2$$

(c) Generalizations:

<u>Type I</u>

$$F^4_{n+2r+1} + L^4_{2r+1}F^4_n + F^4_{n-2r-1} = 2[F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1}]^2$$
$$L^4_{n+2r+1} + L^4_{2r+1}L^4_n + L^4_{n-2r-1} = 2[L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2r-1}]^2$$
$$F^4_{n+2t} + L^4_{2t}F^4_n + F^4_{n-2t} = 2[F^2_{n+2t} + L_{2t}F_nF_{n-2t}]^2$$
$$L^4_{n+2t} + L^4_{2t}L^4_n + L^4_{n-2t} = 2[L^2_{n+2t} + L_{2t}L_nL_{n-2t}]^2$$

<u>Type II</u>

$$F^4_{n+r} + L^4_rF^4 + F^4_{n-r} = 2[F^2_{n+r} + L_rF_nF_{n-r}]^2$$
$$L^4_{n+r} + L^4_rL^4 + L^4_{n-r} = 2[L^2_{n+r} + L_rL_nL_{n-r}]^2$$

<u>Type III</u>

$$F^4_{n+2r+1} + 16L^4_{2r+1}F^4_n + F^4_{n-2r-1} = 2[F^2_{n+2r+1} - 2L_{2r+1}F_nF_{n-2r-1}]^2$$
$$L^4_{n+2r+1} + 16L^4_{2r+1}L^4_n + L^4_{n-2r-1} = 2[L^2_{n+2r+1} - 2L_{2r+1}L_nL_{n-2r-1}]^2$$
$$F^4_{n+2t} + 16L^4_{2t}F^4_n + F^4_{n-2t} = 2[F^2_{n+2t} + 2L_{2t}F_nF_{n-2t}]^2$$
$$L^4_{n+2t} + 16L^4_{2t}L^4_n + L^4_{n-2t} = 2[L^2_{n+2t} + 2L_{2t}L_nL_{n-2t}]^2$$

<u>Type IV</u>

$$F^4_{n+r} + 16L^4_rF^4_n + F^4_{n-r} = 2[F^2_{n+r} + 2L_rF_nF_{n-r}]^2$$
$$L^4_{n+r} + 16L^4_rL^4_n + L^4_{n-r} = 2[L^2_{n+r} + 2L_rL_nL_{n-r}]^2$$

13. (a) Fibonacci-Lucas identity used:

$$F^5_{n+1} - F^5_n - F^5_{n-1} = 5F_nF_{n-1}F_{n+1}(F^2_{n+1} - F_{n-1}F_n)$$

(b) Type I extension:

$$F^5_{n+1} - a^5F^5_n - F^5_{n-1} = 5aF_nF_{n-1}F_{n+1}(F^2_{n+1} - aF_{n-1}F_n)$$

(c) Generalizations:

<u>Type I</u>

$$F^5_{n+2r+1} - L^5_{2r+1}F^5_n - F^5_{n-2r-1} = 5L_{2r+1}F_nF_{n-2r-1}F_{n+2r+1}(F^2_{n+2r+1} - L_{2r+1}F_nF_{n-2r-1})$$
$$L^5_{n+2r+1} - L^5_{2r+1}L^5_n - L^5_{n-2r-1} = 5L_{2r+1}L_nL_{n-2r-1}L_{n+2r+1}(L^2_{n+2r+1} - L_{2r+1}L_nL_{n-2\quad 1})$$
$$L^5_{2t}F^5_n - F^5_{n+2t} - F^5_{n-2t} = 5L_{2t}F_nF_{n-2t}F_{n+2t}(F^2_{n+2t} + L_{2t}F_nF_{n-2t})$$
$$L^5_{2t}L^5_n - L^5_{n+2t} - L^5_{n-2t} = 5L_{2t}L_nL_{n-2t}L_{n+2t}(L^2_{n+2t} + L_{2t}L_nL_{n-2t})$$

<u>Type II</u>

$$L^5_rF^5_n - F^5_{n+r} - F^5_{n-r} = 5L_rF_nF_{n-}\ F_{n+r}(F^2_{n+r} + L_rF_nF_{n-r})$$
$$L^5_rL^5_n - L^5_{n+r} - L^5_{n-r} = 5L_rL_nL_{n-}\ L_{n+r}(L^2_{n+r} + L_rL_nL_{n-r})$$

<u>Type III</u>

$$F^5_{n+2r+1} - 32L^5_{2r+1}F^5_n - F^5_{n-2r-1} = 10L_{2r+1}F_nF_{n-2r-1}F_{n+2r+1}(F^2_{n+2r+1} - 2L_{2r+1}F_nF_{n-2r-1})$$
$$L^5_{n+2r+1} - 32L^5_{2r+1}L^5_n - L^5_{n-2r-1} = 10L_{2r+1}L_nL_{n-2r-1}L_{n+2r+1}(L^2_{n+2r+1} - 2L_{2r+1}L_nL_{n-2r-1})$$

$$32L_{2t}^5F_n^5 - F_{n+2t}^5 - F_{n-2t}^5 = 10L_{2t}F_nF_{n-2t}F_{n+2t}(F_{n+2t}^2 + 2L_{2t}F_nF_{n-2t})$$
$$32L_{2t}^5L_n^5 - L_{n+2t}^5 - L_{n-2t}^5 = 10L_{2t}L_nL_{n-2t}L_{n+2t}(L_{n+2t}^2 + 2L_{2t}L_nL_{n-2t})$$

Type IV

$$32L_r^5F_n^5 - F_{n+r}^5 - F_{n-r}^5 = 10L_rF_nF_{n-r}F_{n+r}(F_{n+r}^2 + 2L_rF_nF_{n-r})$$
$$32L_r^5L_n^5 - L_{n+r}^5 - L_{n-r}^5 = 10L_rL_nL_{n-r}L_{n+r}(L_{n+r}^2 + 2L_rL_nL_{n-r})$$

14. (a) Fibonacci-Lucas identity used: $L_n^3 = 2F_{n-1}^3 + F_n^3 + 6F_{n+1}^2F_{n-1}$
(b) Type I extension: $b^3L_n^3 = 2F_{n-1}^3 + a^3F_n^3 + 6F_{n+1}^2F_{n-1}$
(c) Generalizations:

Type I

$$F_{2r+1}^3L_n^3 = 2F_{n-2r-1}^3 + L_{2r+1}^3F_n^3 + 6F_{n+2r+1}^2F_{n-2r-1}$$
$$D^3F_{2r+1}^3F_n^3 = 2L_{n-2r-1}^3 + L_{2r+1}^3L_n^3 + 6L_{n+2r+1}^2L_{n-2r-1}$$
$$F_{2r}^3L_n^3 = L_{2r}^3F_n^3 - 2F_{n-2r}^3 - 6F_{n+2r}^2F_{n-2r}$$
$$D^3F_{2r}^3F_n^3 = L_{2r}^3L_n^3 - 2L_{n-2r}^3 - 6L_{n+2r}^2L_{n-2r}$$

Type II

$$F_r^3L_n^3 = L_r^3F_n^3 - 2F_{n-r}^3 - 6F_{n+r}^2F_{n-r}$$
$$D^3F_r^3F_n^3 = L_r^3L_n^3 - 2L_{n-r}^3 - 6L_{n+r}^2L_{n-r}$$

Type III

$$4F_{2r+1}^3L_n^3 = F_{n-2r-1}^3 + 4L_{2r+1}^3F_n^3 + 3F_{n+2r+1}^2F_{n-2r-1}$$
$$4D^3F_{2r+1}^3F_n^3 = L_{n-2r-1}^3 + 4L_{2r+1}^3L_n^3 + 3L_{n+2r+1}^2L_{n-2r-1}$$
$$4F_{2r}^3L_n^3 = 4L_{2r}^3F_n^3 - F_{n-2r}^3 - 3F_{n+2r}^2F_{n-2r}$$
$$4D^3F_{2r}^3F_n^3 = 4L_{2r}^3L_n^3 - L_{n-2r}^3 - 3L_{n+2r}^2L_{n-2r}$$

Type IV

$$4F_r^3L_n^3 = 4L_r^3F_n^3 - F_{n-r}^3 - 3F_{n+r}^2F_{n-r}$$
$$4D^3F_r^3F_n^3 = 4L_r^3L_n^3 - L_{n-r}^3 - 3L_{n+r}^2L_{n-r}$$

Concluding Remarks

Following the suggestions of the referee and the editor, the proofs of the 14 identity sets have been omitted. They are tedious and do involve complicated, albeit fairly elementary, calculations. For some readers, the proofs would involve the use of composition algebras which are not developed in the article and which may not be well known.

The author has completed a supplementary paper giving, with indicated proof, the Type I, Type II, Type III, and Type IV composition algebras. After each composition albegra the corresponding identities using that algebra have been stated and proved. Copies of this paper may be obtained by request from the author.

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A FORMULA FOR TRIBONACCI NUMBERS

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In a recent paper [2], Scott, Delaney, and Hoggatt discussed the Tribonacci numbers T_n defined by

$$T_0 = 1,\ T_1 = 1,\ T_2 = 2 \quad \text{and} \quad T_n = T_{n-1} + T_{n-2} + T_{n-3},\ \text{for } n \geq 3,$$