A NOTE ON FIBONACCI NUMBERS

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We shall employ the notation

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = u_n + u_{n-1} \quad (n \geq 1) ,$$
$$v_0 = 2, \quad v_1 = 1, \quad v_{n+1} = v_n + v_{n-1} \quad (n \geq 1) .$$

Thus

(1) $$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n ,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad \alpha + \beta = 1, \quad \alpha\beta = -1 .$$

The first few values of u_n, v_n follow.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
u_n	0	1	1	2	3	5	8	13	21	34	55	89	144
v_n	2	1	3	4	7	11	18	29	47	76	123	199	322

It follows easily from the definition of (1) that

(2) $$u_n = u_{n-k+1}u_k + u_{n-k}u_{k-1} \quad (n \geq k \geq 1) ,$$

(3) $$v_n = u_{n-k+1}v_k + u_{n-k}v_{k-1} \quad (n \geq k \geq 1) .$$

It is an immediate consequence of (1) that

(4) $$u_k \mid u_{mk} ,$$

(5) $$v_k \mid u_{2mk} ,$$

(6) $$v_k \mid v_{(2m-1)k} ,$$

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where m and k are arbitrary positive integers. It is perhaps not so familiar that, conversely,

(4)' $$u_k \mid u_n \Longrightarrow n = mk \qquad (k > 2) \quad ,$$

(5)' $$u_k \mid u_n \Longrightarrow n = 2mk \qquad (k > 1) \quad ,$$

(6)' $$v_k \mid v_n \Longrightarrow n = (2m - 1)k \quad (k > 1) \quad .$$

These results can be proved rapidly by means of (1) and some simple results about algebraic numbers. If we put

(7) $$n = mk + r \qquad (0 \le r < k) \quad ,$$

then

$$\alpha^n - \beta^n = \alpha^r(\alpha^{mk} - \beta^{mk}) + \beta^{mk}(\alpha^r - \beta^r) \quad ,$$

so that

$$u_n = \alpha^r u_{mk} + \beta^{mk} u_r \quad .$$

If $u_k \mid u_n$ it therefore follows that $u_k \mid \beta^{mk} u_r$. Since β is a unit of the field $R(\sqrt{5})$, $u_k \mid u_r$, which requires $r = 0$. This proves (4)'.

Similarly if

$$n = 2mk + r \quad (0 \le r < 2k) \quad ,$$

then

$$u_n = \alpha^r u_{2mk} + \beta^{2mk} u_r \quad .$$

Hence if $v_k \mid u_n$ it follows that $v_k \mid u_r$. If then $r > 0$ we must have $r > k$ and the identity

$$(\alpha - \beta) u_r = \alpha^{r-k} v_k - \beta v_{r-k}$$

gives $v_k \mid v_{r-k}$, which is impossible. The proof of (6)' is similar.

If we prefer, we can prove (4)', (5)', (6)' without reference to algebraic numbers. For example if $u_k \mid u_n$, then (2) implies $u_k \mid u_{n-k}u_{k-1}$. Since u_k and u_{k-1} are relatively prime we have $u_k \mid u_{n-k}$. Continuing in this way we get $u_k \mid u_r$, where r is defined by (7). The proof is now completed as above. In the same way we can prove (5)' and (6)'.

In view of the relation

$$u_{2n} = u_n v_n \tag{8}$$

it is natural to ask for the general solution of the equation

$$u_n = u_m v_k \quad (m > 2,\ k > 1) . \tag{9}$$

It is easily verified, using (1), that (9) can be replaced by

$$u_n = u_{m+k} + (-1)^k u_{m-k} \quad (m \geq k) \tag{10}$$

or

$$u_n = u_{m+k} - (-1)^k u_{k-m} \quad (k > m) . \tag{11}$$

Now the equation

$$u_r = u_s + u_t \quad (s > t > 1) \tag{12}$$

is satisfied only when $r - 1 = s = t + 1$. Indeed if $1 < t < s - 1$, then

$$u_s + u_t < u_s + u_{s-1} = u_{s+1} ,$$

so that (12) is impossible; if $t = s - 1$, then clearly $r = s + 1$. If $t = 1$ in (12) we have the additional solution $r = 4,\ s = 3$.

Returning to (10) and (11) we first dispose of the case $m - k = 1$. For k even (10) will be satisfied only if $m + k = 3$, which implies $k = 1$; for k odd we get $n = 2,\ m + k = 3$ or $n = 3,\ m + k = 4$, which is impossible. Equation (11) with $k - m = 1$ is disposed of in the same way.

We may therefore assume in (10) and (11) that $|m-k| > 1$. Then if k is even, it is evident from the remark concerning (12) that (10) is impossible. If k is odd, we have

$$u_{m+k} = u_n + u_{m-k} ,$$

so that $k = 1$, $m = n$. As for (11), if m is odd we get

$$u_n = u_{m+k} + u_{k-m} ,$$

which is impossible. However, if m is even, we get

$$u_{m+k} = u_n + u_{k-m} ,$$

so that $m + k = n + 1 = k - m + 2$; this requires $m = 1$, $k = n$.

This completes the proof of

Theorem 1. The equation

$$u_n = u_m v_k \quad (m > 2,\ k > 1)$$

has only the solutions $n = 2m = 2k$.

The last part of the above proof suggests consideration of the equation

$$u_n = v_k \quad (k > 1) . \tag{13}$$

Since (13) is equivalent to

$$u_n = u_{k+1} + u_{k-1} ,$$

it follows at once that the only solution of (13) is $n = 4$, $k = 2$.

The equation

$$u_n = v_m v_k \quad (m \geq k > 1) , \tag{14}$$

is equivalent to

$$u_n = v_{m+k} + (-1)^k v_{m-k} . \tag{15}$$

If k is even it is clear that $n > m+k$; indeed since $v_{m+k} = u_{m+k+1} + u_{m+k-1}$ we must have $n > m + k + 1$. Then (15) implies

$$u_{m+k+2} \leq u_{m+k+1} + u_{m+k-1} + v_{m-k} ,$$

which simplifies to

$$(16) \qquad u_{m+k-2} \leq v_{m-k} .$$

If $m = k$, (16) holds only when $m = 2$; however this does not lead to a solution of (14). If $m > k$, (16) may be written as

$$u_{m+k-2} \leq u_{m-k+1} + u_{m-k-1} < u_{m-k} ,$$

which holds only when $m = 4$, $k = 2$.

If k is odd, (15) becomes

$$(17) \qquad u_n + v_{m-k} = v_{m+k} .$$

If $m = k$ this reduces to

$$u_n + 2 = u_{2k+1} + u_{2k-1} ,$$

which implies $2k - 1 = 3$, $k = 2$. If $m = k + 1$ (17) gives

$$u_n + 1 = u_{2k+2} + u_{2k} ,$$

which is clearly impossible. For $m > k + 1$ we get

$$u_{m+k+1} + u_{m+k-1} \geq u_n + 2u_{m-k} ,$$

so that $n \leq m + k + 1$. Since

$$u_{m+k} + 2u_{m-k} < u_{m+k+1} + u_{m+k-1} ,$$

we must have $n = m + k + 1$. Hence (17) becomes

$$v_{m-k} = u_{m+k-1} ;$$

as we have seen above, this implies

$$m - k = 2, \quad m + k - 1 = 4 ,$$

so that we do not get a solution.

We may state

Theorem 2. The equation

$$u_n = v_m v_k \quad (m \geq k > 1)$$

has the unique solution $n = 8$, $m = 4$, $k = 2$.

It is clear from (4)' that the equation

$$u_n = cu_k \quad (k > 2) , \tag{18}$$

where c is a fixed integer >1 is solvable only when $k|n$. Moreover the number of solutions is finite. Indeed (18) implies

$$cu_k \geq u_{2k} \geq u_k v_k , \quad c \geq v_k ;$$

moreover if $n = rk$ then for fixed k, r is uniquely determined by (18).

This observation suggests two questions: For what values of c is (18) solvable and, secondly, can the number of solutions exceed one? In connection with the first question consider the equation

$$u_n = 2u_k \quad (k > 2) . \tag{19}$$

Since for $n > 3$

$$2u_{n-2} < u_n = 2u_{n-2} + u_{n-3} < 2u_{n-1} ,$$

we get

$$u_{n-2} < u_k < u_{n-1} ,$$

which is clearly impossible. Similarly, since for $n > 4$

$$3u_{n-3} < u_n = 3u_{n-3} + 2u_{n-4} < 3u_{n-2} ,$$

it follows that the equation

(20) $$u_n = 3u_k \quad (k > 2)$$

has no solution.

Let us consider the equation

(21) $$u_n = u_m u_k \quad (m \geq k > 2) .$$

We take

$$u_n = u_{n-m+1}u_m + u_{n-m}u_{m-1} ,$$

so that

$$u_{n-m+1}u_m < u_n < u_{n-m+2}u_m ,$$

provided $n > m$. Then clearly (21) is impossible.

For the equation

(22) $$v_n = u_m v_k \quad (m > 2,\ k > 1) ,$$

we use

$$v_n = u_m v_{n-m+1} + u_{m-1}v_{n-m} .$$

Then

$$u_m v_{n-m+1} < v_n < u_m v_{n-m+2} ,$$

so that (22) is impossible.

This proves

Theorem 3. Each of the equations (21), (22) possesses no solutions.

Consider next the equation

$$(23) \qquad v_n = v_m v_k \quad (m \geq k > 1) .$$

This is equivalent to

$$(24) \qquad v_n = v_{m+k} + (-1)^k v_{m-k} .$$

For k even, (24) is obviously impossible. For k odd we may write

$$v_{m+k} = v_n + v_{m-k} ,$$

which requires $m + k = n + 1 = m - k + 2$, so that $k = 1$. This proves

Theorem 4. The equation (23) possesses no solutions.

The remaining type of equation is

$$(25) \qquad v_n = u_m u_k \quad (m \geq k > 2) .$$

This is equivalent to

$$(26) \qquad 5v_n = v_{m+k} + (-1)^k v_{m-k} .$$

Clearly $n < m + k$. Then since

$$v_{m+k} = 5v_{m+k-4} + 3v_{m+k-5} ,$$

(26) implies

$$(27) \qquad 5v_n = 5v_{m+k-4} + 3v_{m+k-5} + (-1)^k v_{m-k} .$$

Consequently $n \geq m + k - 3$, while the right member of (27) is less than

$$5v_{m+k-4} + 4v_{m+k-5} < 5v_{m+k-3} .$$

This evidently proves

Theorem 5. The equation (25) possesses no solution.

Next we discuss the equations

$$u_m^2 + u_n^2 = u_k^2 \quad (0 < m \leq n) \quad , \tag{28}$$

$$v_m^2 + v_n^2 = v_k^2 \quad (0 \leq m \leq n) \quad . \tag{29}$$

We shall require the following

Lemma. The following inequalities hold.

$$\frac{u_{n+1}}{u_n} \geq \frac{3}{2} \quad (n \geq 2) \quad , \tag{30}$$

$$\frac{v_{n+1}}{v_n} \geq \frac{3}{2} \quad (n \geq 3) \quad . \tag{31}$$

Proof. Since $u_n \leq 2u_{n-1}$ for $n \geq 2$, we have

$$\frac{u_{n+1}}{u_n} = 1 + \frac{u_{n-1}}{u_n} \geq \frac{3}{2} \quad .$$

The proof of (31) is exactly the same.

Returning to (28) it is evident that

$$u_n^2 < u_k^2 < 2u_n^2 \quad ,$$

so that

$$u_n < u_k < u_n\sqrt{2} \quad .$$

Then $k > n$ and by the lemma

$$u_k \geq u_{n+1} \geq \frac{3}{2} u_n \quad .$$

Since $\sqrt{2} < 3/2$, we have a contradiction. The same argument applies to (29). The lemma requires that $n \geq 2$ or 3 but there is of course no difficulty about the excluded values. This proves

Theorem 6. Each of the equations (28), (29), possesses no solutions. More generally, each of the equations

$$u_m^r + u_n^r = u_k^r \quad (0 < m \le n) ,$$

$$v_m^r + v_n^r = v_k^r \quad (0 \le m \le n) ,$$

where $r \ge 2$ has no solutions.

Remark. The impossibility of (29) can also be inferred rapidly from the easily proved fact that no v_n is divisible by 5. Indeed since

$$\alpha^5 \equiv \beta^5 \equiv \frac{1}{2} \quad (\mathrm{mod}\ \sqrt{5}) ,$$

it follows that

$$v_{n+5} = \alpha^{n+5} + \beta^{n+5} \equiv \frac{1}{2}(\alpha^n + \beta^n) = \frac{1}{2}v_n \quad (\mathrm{mod}\ \sqrt{5}) ,$$

so that $v_{n+5} \equiv v_m (\mathrm{mod} \sqrt{5})$. Moreover none of v_0, v_1, v_3, v_4 is divisible by 5.

The mixed equation

$$v_m^2 + v_n^2 = u_k^2 \quad (0 \le m \le n) \tag{32}$$

has the obvious solution $m = 2$, $n = 3$, $k = 5$; the equation

$$u_m^2 + v_n^2 = u_k^2 \quad (m > 0) \tag{33}$$

has the solution $m = 4$, $n = 3$, $k = 5$.

Clearly (32) implies

$$v_n < u_k < v_n\sqrt{2} .$$

This inequality is not sufficiently sharp to show that (32) has no solutions although it does suffice for the equation

$$v_m^r + v_n^r = u_k^r$$

with r sufficiently large.

However (32) is equivalent to

$$v_{2m} + (-1)^m 2 + v_{2n} + (-1)^n 2 = \frac{1}{5}\{v_{2k} - (-1)^k 2\} \quad . \tag{34}$$

If $m + n \equiv 1 \pmod 2$, this reduces to

$$v_{2m} = v_{2n} = \frac{1}{5}\{v_{2k} - (-1)^k 2\} \quad .$$

There is no loss in generality in assuming $k \geq 5$. Then since

$$v_{2k} = 5v_{2k-4} + 3v_{2k-5} \quad ,$$

we get

$$v_{2m} + v_{2n} = v_{2k-4} + \frac{1}{5}\{3v_{2k-5} - (-1)^k 2\} \quad .$$

Since $m < n$ and

$$\frac{1}{5}\{3v_{2k-5} - (-1)^k 2\} < v_{2k-5} \quad ,$$

we must have $2n = 2k - 4$ and

$$5v_{2m} = 3v_{2k-5} - (-1)^k 2 = 6v_{2k-7} + 3v_{2k-8} - (-1)^k 2 \quad .$$

It is therefore necessary that $2m = 2k - 6$ and we get

$$5v_{2m} = 6v_{2m-1} + 3v_{2m-2} + (-1)^m 2 \quad ,$$

which simplifies to

$$v_{2m-4} = (-1)^m 2 \quad .$$

Hence $m = 2$, $k = 5$, $n = 3$ (a solution of (22)).

Next if $m \equiv n \pmod 2$, (34) reduces to

$$v_{2m} + v_{2n} + (-1)^n 4 = \frac{1}{5}\{v_{2k} - (-1)^k 2\}$$

and as above we get

$$v_{2m} + v_{2n} + (-1)^m 4 = v_{2k-4} + \frac{1}{5}\{3v_{2k-5} - (-1)^k 2\} . \tag{35}$$

It is necessary that $2n = 2k - 4$, so that (35) reduces to

$$5v_{2m} + (-1)^m 20 = 3v_{2k-5} - (-1)^k 2 . \tag{36}$$

Clearly $2m \le 2k - 6$. If $2m < 2k - 6$ we get

$$3v_{2k-5} - (-1)^k 2 \le 5v_{2k-7} + (-1)^m 20 ,$$

or

$$v_{2k-6} + 2v_{2k-8} \le (-1)^m 20 + (-1)^k 2 ,$$

which is not possible. Thus $2m = 2k - 6$ and (36) becomes

$$5v_{2m} + (-1)^m 20 = 3v_{2m+1} + (-1)^m 2 .$$

This reduces to

$$v_{2m-4} = (-1)^{m-1} 18 ,$$

which is satisfied by $m = 5$. Then $k = 8$, $n = 6$ but this does not lead to a solution of (32).

This completes the proof of

Theorem 7. The equation

$$v_m^2 + v_n^2 = u_k^2 \quad (0 \le m \le n)$$

has the unique solution $m = 2$, $n = 3$, $k = 5$.

The equation

(37) $$u_m^2 + v_n^2 = u_k^2 \quad (m > 0)$$

can be treated in a less tedious manner. Suppose first that $v_n < u_m$. Then (37) implies

$$u_m^2 < u_k^2 < 2u_m^2$$

and as we have seen above this is impossible. Next let $u_m < v_n$. If $k > n + 2$ then

$$u_k^2 \geq u_{n+3}^2 = (2u_{n+1} + u_n)^2 = 2(u_{n+1} + u_{n-1})^2 + 2u_{n+1}^2 + 2u_{n+1}u_{n-2} + u_n^2 - u_{n-1}^2 > 2v_n^2 ,$$

so that (37) is certainly not satisfied. Since $k > n + 1$ it follows that $k = n + 2$. Thus (37) becomes

(38) $$u_m^2 = u_{n+2}^2 - v_n^2 = 3(u_n^2 - u_{n-1}^2)$$

as is easily verified. If $m > n + 2$ then

$$u_m^2 \geq u_{n+2}^2 = (2u_n + u_{n-1})^2 > 3(u_n^2 - u_{n-1}^2) ,$$

contradicting (38). Since for $n > 3$

$$3(u_n^2 - u_{n-1}^2) - u_n^2 = 2u_n^2 - 3u_{n-1}^2 > \frac{9}{2}u_{n-1}^2 - 3u_{n-1}^2 > 0$$

it follows that $m > n$. Thus $m = n + 1$ and (38) becomes

$$u_{n+1}^2 = 3(u_n^2 - u_{n-1}^2) .$$

This implies $u_n + u_{n-1} = 3$, $n = 3$, which leads to the solution $n = 3$, $m = 4$, $k = 5$ of (37). As for the excluded values $n = 1, 2$ it is obvious that they do not furnish a solution. This proves

Theorem 8. The equation

$$u_m^2 + v_n^2 = u_k^2 \quad (m > 0)$$

has the unique solution $m = 4$, $n = 3$, $k = 5$.

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