LATTICE PATHS AND FIBONACCI NUMBERS

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L. Moser and W. Zayachkowski [1] considered lattice paths from $(0,0)$ to (m,n) where the possible moves were of three types: (i) horizontal moves from (x,y) to $(x+1,y)$; (ii) vertical moves from (x,y) to $(x,y+1)$, and (iii) diagonal moves from (x,y) to $(x+1,y+1)$. A special case of some interest arises when $m = n$.

Consider now a much more restricted set of paths. Require: (a) that the path be symmetric about the line $x+y = n$, (b) that prior to arriving or touching the above line that one use only horizontal and diagonal moves (and symmetry after now requires that vertical and diagonal moves be used to arrive at (n,n)), and (c) that all of the paths be "below" or on the line $y = x$ (also required by the previous conditions).

For small values of n one can enumerate the possible paths. Thus for $n = 1$, one need only consider the three points $(0,0)$, $(1,0)$ and $(1,1)$, and there are two paths as pictured in Fig. 1. For $n = 2$, there will be three paths. For $n = 3$ there will be five paths. See Figs. 2 and 3, respectively.

This suggests that the collection of path numbers may be closely related to the Fibonacci sequence, with appropriate renumbering to bring the two sequences into step. Thus, letting $h(n)$ be the number of paths for $(0,0)$, (n,n) case, one has the tabulation

n	1	2	3
h(n)	2	3	5

Also, beginning at $(0,0)$, there are only two initial moves, to $(1,0)$ and to $(1,1)$. Due to symmetry imposed by requirement (a) the last move in the path is also determined so that one has the choices schematically portrayed in Fig. 4. The two path schemes depicted in Fig. 4 are mutually exclusive and collectively they exhaust all of the allowable paths satisfying conditions (a), (b) and (c). Hence, $h(n) = h(n-1) + h(n-2)$ for $n = 3, 4, \cdots$. Thus the sequence of path numbers is a Fibonacci sequence with appropriate relabelling and identification of $h(0)$ and $h(-1)$ as unity.

It is also possible to "count" the paths so as to have

$$h(n) = \sum_{k=0} \binom{n+1-k}{k}$$

but the grouping of paths with summation index k seems slightly artificial and lends little or nothing to the general theory.

Figure 1 (two cases)

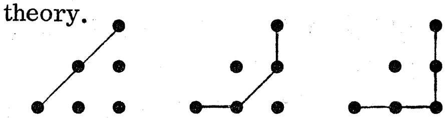


Figure 2 (three cases)

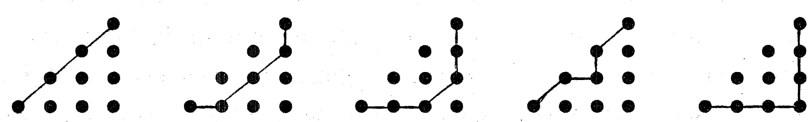
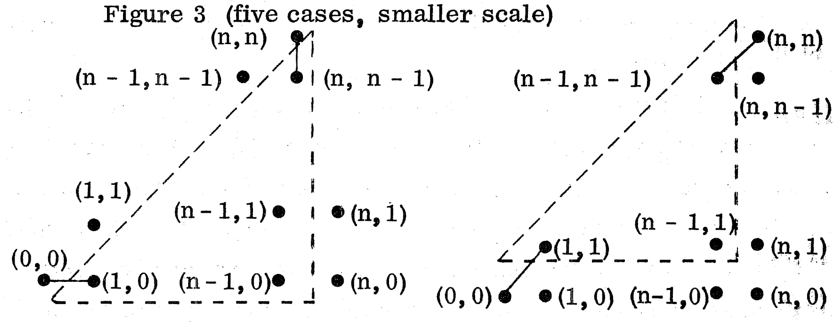


Figure 3 (five cases, smaller scale)



There are h(n - 1) paths from (1, 0) to (n, n - 1) inside dashed triangle.

There are h(n - 2) paths from (1, 1) to (n - 1, n - 1) insided dashed triangle.

Figure 4

REFERENCE

1. L. Moser and W. Zayachkowski, "Lattice Paths with Diagonal Steps," Scripta Mathematica, Vol. XXVI, No. 3, pp. 223–229.