A PRIMER FOR THE FIBONACCI NUMBERS – PART V

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CORRECTION

Read the last displayed equation, on page 67 of Part IV, as

$$\operatorname{Tan}\left\{\operatorname{Tan}^{-1}\frac{F_n}{F_{n+1}} - \operatorname{Tan}^{-1}\frac{\sqrt{5}-1}{2}\right\} = (-1)^{n+1}\left(\frac{\sqrt{5}-1}{2}\right)^{2n+1}$$

1. INTRODUCTION

In Section 8 of Part IV, we discussed an alternating series. This time we shall lay down some brief foundations of sequences and infinite series. This leads to some very interesting results in this issue and to the broad topic of generating functions in the next issue and to continued fractions in the issue after that. Many Fibonacci numbers shall appear.

2. SEQUENCES

Definition: An ordered set of numbers $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an infinite sequence of numbers. If there are but a finite number of the a's, $a_1, a_2, \cdots, a_n$ then it is a finite sequence of numbers.

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to have a real number, a, as a limit (written $\lim_{n\to\infty} a_n = a$) if for every positive real number ϵ, $|a_n - a| < \epsilon$ for all but a finite number of the members of the sequence $\{a_n\}$. If the sequence $\{a_n\}$ has a limit, this limit is unique and the sequence is said to converge to this limit. If the sequence $\{a_n\}$ fails to approach a limit, then the sequence is said to diverge. We now give examples of each kind.

If $a_n = 1$, $\{a_n\} = 1, 1, 1, \cdots$ converges since $\lim_{n\to\infty} a_n = 1$.

If $a_n = 1/n$, $\{a_n\} = 1, 1/2, 1/3, \cdots, 1/n, \cdots$ converges to zero.

If $a_n = (-1)^n$, $\{a_n\} = 1, -1, +1, -1, +1, \cdots$ diverges by oscillation. That is, it does not approach any limit.

If $a_n = n$, $\{a_n\} = 1, 2, 3, \cdots$ diverges to plus infinity.

Finally if $a_n = \frac{n}{n+1}$, then $\{a_n\} = \frac{1}{2}, \frac{2}{3}, \cdots$ converges to one.

Some limit theorems for sequences are the following:

If $\{a_n\}$ and $\{b_n\}$ are two sequences of real numbers with limits a and b, respectively, then

$$\lim_{n\to\infty} (a_n + b_n) = a + b$$

$$\lim_{n\to\infty} (a_n - b_n) = a - b$$

$$\lim_{n\to\infty} (c\, a_n) = ca, \quad \text{any real } c$$

$$\lim_{n\to\infty} a_n b_n = ab$$

$$\lim_{n\to\infty} (a_n / b_n) = a/b, \quad b \neq 0 \ .$$

3. BOUNDED MONOTONE SEQUENCES

The sequence $\{a_n\}$ is said to be bounded if there exists a positive number, K, such that $|a_n| < K$ for all $n \geq 1$. If $a_{n+1} \geq a_n$, for $n \geq 1$, the sequence $\{a_n\}$ is said to be a monotone increasing sequence; if $a_n \geq a_{n+1}$ for $n \geq 1$, the sequence is monotone decreasing sequence. If a sequence is such that it is either monotone increasing or monotone decreasing it will be called a monotone sequence.

The following useful and important theorem is stated without proof:

Theorem 1: A bounded monotone sequence converges.

As an example, consider the sequence $\{(1 + 1/n)^n\}$, this sequence is monotone increasing and bounded above by 3. The limit of this sequence is well known. We will use Theorem 1 in the material to come.

4. ANOTHER IMPORTANT THEOREM

The following sufficient conditions for the convergence of an alternating series are given below.

Theorem 2: If, for the sequence $\{s_n\}$,

1. $S_1 > 0$,
2. $(S_{n-1} - S_n)(-1)^n > (S_n - S_{n+1})(-1)^{n+1} > 0$, for $n \geq 2$,

3. $\lim_{n \to \infty} (S_n - S_{n+1}) = 0$,

then the sequence $\{S_n\}$ converges to a limit, S, such that $0 < S < S_1$.

5. AN EXAMPLE OF AN APPLICATION OF THEOREM 2

For the following example a limit is known to exist by the application of Theorem 2 of Section 4.

Let $S_n = F_n / F_{n+1}$, where $\{F_n\}$ is the Fibonacci sequence, then $S_{n-1} - S_n = (-1)^n / (F_n F_{n+1})$. By Theorem 2 above, $\lim_{n \to \infty} S_n$ exists.

To find the limit, consider

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} \quad ,$$

which in terms of $\{S_n\}$ is $1/S_n = 1 + S_{n-1}$. Let the limit of S_n as n tends to infinity be S, then $\lim_{n \to \infty} S_n = \lim_{n \to \infty} S_{n-1} = S > 0$. Applying the limit theorems of Section 2, it follows that S satisfies

$$S = \frac{1}{1 + S} \quad \text{or} \quad S^2 + S - 1 = 0$$

Thus $S > 0$ is given by

$$S = \frac{\sqrt{5} - 1}{2}$$

the positive root of the quadratic equation $S^2 + S - 1 = 0$.

6. INFINITE SERIES

If we add together the members of a sequence $\{a_n\}$, we get the infinite series $a_1 + a_2 + \cdots + a_n + \cdots$. We now get another sequence from this infinite series.

Define a sequence $\{S_n\}$ in the following way. Let $S_1 = a_1 = \sum_{i=1}^{1} a_i$, $S_2 = a_1 + a_2 = \sum_{i=1}^{2} a_i, \cdots$ or in general $S_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$. This is called the sequence of partial sums of the infinite series. The infinite series can also be denoted by

$$A = a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{i=1}^{\infty} a_i \quad .$$

If the sequence $\{S_n\}$ converges to a limit, S, then the infinite series, A, is said to converge and converge to the limit S; otherwise series A is said to diverge.

7. SPECIAL RESULTS CONCERNING SERIES

1. If an infinite series $A = a_1 + a_2 + \cdots + a_n + \cdots$ converges, then $\lim_{n\to\infty} a_n = 0$. This is immediate since $a_n = S_n - S_{n-1}$.

2. From Section 3 above, an infinite series of positive terms converges if the partial sums are bounded above since the partial sums form a monotone increasing sequence.

3. For the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad \text{such that} \quad a_n > 0,\ n \geq 1;\ a_{n+1} \leq a_n,\ n \geq 1;\ \lim_{n\to\infty} a_n = 0$$

then by Section 4, above, the infinite series converges; in the theorem

$$S_n = \sum_{j=1}^{n} (-1)^j a_j \quad .$$

An example of an alternating series was seen in Part IV, Section 8, of this Primer.

8. FIBONACCI NUMBERS, LUCAS NUMBERS AND Π

It is well known and easily verified that

$$\frac{\Pi}{4} = \mathrm{Tan}^{-1}\frac{1}{1} = \mathrm{Tan}^{-1}\frac{1}{2} + \mathrm{Tan}^{-1}\frac{1}{3} \quad .$$

Also one can verify

$$\frac{\Pi}{4} = \operatorname{Tan}^{-1}\frac{1}{1} = \operatorname{Tan}^{-1}\frac{1}{2} + \operatorname{Tan}^{-1}\frac{1}{5} + \operatorname{Tan}^{-1}\frac{1}{8}$$

or

$$\frac{\Pi}{4} = \operatorname{Tan}^{-1}\frac{1}{3} + \operatorname{Tan}^{-1}\frac{1}{5} + \operatorname{Tan}^{-1}\frac{1}{7} + \operatorname{Tan}^{-1}\frac{1}{8}$$

We note Fibonacci and Lucas numbers here, surely. We shall here easily extend these results in several ways.

In this section we shall use several new identities which are left as exercises for the reader and will be marked with an asterisk.

*<u>Lemma 1</u>: $L_{2n} L_{2n+2} - 1 = 5F_{2n+1}^2$. This is really a special case of a generalization of B-22, p. 76, Oct., 1963, <u>Fibonacci Quarterly</u>.

<u>Lemma 2</u>: $L_n^2 = L_{2n} + 2(-1)^n$

<u>Lemma 3</u>: $L_n^2 - 5F_n^2 = 4(-1)^n$

*<u>Lemma 4</u>: $L_n L_{n+1} = L_{2n+1} + (-1)^n$

We now discuss

<u>Theorem 3</u>: If $\tan \psi_n = 1/L_n$, then

$$\operatorname{Tan}(\psi_{2n} + \psi_{2n+2}) = 1/F_{2n+1} \text{ or } \operatorname{Tan}^{-1}\frac{1}{F_{2n+1}} = \operatorname{Tan}^{-1}\frac{1}{L_{2n}} + \operatorname{Tan}^{-1}\frac{1}{L_{2n+2}} .$$

<u>Proof</u>:

$$\operatorname{Tan}(\psi_{2n} + \psi_{2n+2}) = \frac{L_{2n} + L_{2n+2}}{L_{2n}L_{2n+2} - 1} = \frac{1}{F_{2n+1}}$$

since

$$L_{2n+2} + L_{2n} = 5F_{2n+1} \text{ and } L_{2n}L_{2n+2} - 1 = 5F_{2n+1}^2$$

by Lemma 1 above.

Theorem 4: If $\tan\theta_n = 1/F_n$, then $\operatorname{Tan}(\theta_{2n} - \theta_{2n+2}) = 1/F_{2n+1}$,

or

$$\text{Tan}^{-1}\frac{1}{F_{2n+1}} = \text{Tan}^{-1}\frac{1}{F_{2n}} - \text{Tan}^{-1}\frac{1}{F_{2n+2}}$$

Proof:

$$\text{Tan}(\theta_{2n} - \theta_{2n+2}) = \frac{F_{2n+2} - F_{2n}}{F_{2n}F_{2n+2} + 1} = \frac{1}{F_{2n+1}}$$

since

$$F_{2n+2} - F_{2n} = F_{2n+1} \quad \text{and} \quad F_{2n}F_{2n+2} - F_{2n+1}^2 = (-1)^{2n+1}$$

From Theorem 4,

$$\sum_{n=1}^{M} \text{Tan}^{-1}\frac{1}{F_{2n+1}} = \sum_{n=1}^{M}\left(\text{Tan}^{-1}\frac{1}{F_{2n}} - \text{Tan}^{-1}\frac{1}{F_{2n+2}}\right)$$

$$= \text{Tan}^{-1}\frac{1}{F_2} - \text{Tan}^{-1}\frac{1}{F_{2M+2}} \; .$$

Since $\lim_{M\to\infty} \text{Tan}^{-1}\frac{1}{F_{2M+2}} = 0$ by continuity of $\text{Tan}^{-1}x$ at $x = 0$ we may write

Theorem 5:

$$\frac{\Pi}{4} = \text{Tan}^{-1} 1 = \sum_{n=1}^{\infty} \text{Tan}^{-1}\frac{1}{F_{2n+1}} \; .$$

This is the celebrated result of D. H. Lehmer, Nov. 1936, American Mathematical Monthly, p. 632, Problem 3801.

We note in passing that the partial sums

$$S_M = \sum_{n=1}^{M} \text{Tan}^{-1}\frac{1}{F_{2n+1}} = \text{Tan}^{-1}\frac{1}{F_2} - \text{Tan}^{-1}\frac{1}{F_{2M+2}}$$

are all bounded above by $\text{Tan}^{-1}1 = \Pi/4$ and S_M is monotone. Thus Theorem 1 can be applied. From Theorem 3,

$$\sum_{n=1}^{M} \text{Tan}^{-1}\frac{1}{F_{2n+1}} = \sum_{n=1}^{M}\left(\text{Tan}^{-1}\frac{1}{L_{2n}} + \text{Tan}^{-1}\frac{1}{L_{2n+2}}\right)$$

so that

$$\sum_{n=1}^{M} \mathrm{Tan}^{-1} \frac{1}{F_{2n+1}} + \mathrm{Tan}^{-1} \frac{1}{3} = 2 \sum_{n=1}^{M} \mathrm{Tan}^{-1} \frac{1}{L_{2n}} + \mathrm{Tan}^{-1} \frac{1}{L_{2M+2}}$$

The limit on the left tends to $\mathrm{Tan}^{-1} 1 + \mathrm{Tan}^{-1} 1/3 = \mathrm{Tan}^{-1} 2$ and the right-hand side tends to this same limit and since $\mathrm{Tan}^{-1} 1/L_{2M+2} \to 0$, then

Theorem 6:

$$\sum_{n=1}^{\infty} \mathrm{Tan}^{-1} \frac{1}{L_{2n}} = \mathrm{Tan}^{-1} \frac{\sqrt{5} - 1}{2} = \frac{1}{2} \mathrm{Tan}^{-1} 2 \quad .$$

Compare with Theorem 5 in Part IV.

We shall continue this interesting discussion in the next issue.

CORRECTIONS FOR VOLUME 1, NO. 2

Page 45: In the tenth line up from the bottom, the subscripts on the Fibonacci numbers should be reversed.

Page 47: Replace "Lamda" by "Lambda" in the title.

Page 52: In line 6, replace (R^n) with $\lambda(R^n)$.
In line 12, the author's name is Jekuthiel Ginsburg.

Page 55: In problem H-18, part a, replace $=$ by $\doteq$.

Page 57: In E2, replace $\frac{a}{d}, \frac{b}{d}$ with $\left(\frac{a}{d}, \frac{b}{d}\right)$.

Page 58: Add three dots after the 4 on the last line.

Page 60: The title "Letters to the Editor" was omitted from Fibonacci Formulas, and, in that article, the "Correct Formula" due to the late Jekuthiel Ginsburg is $F^3_{n+2} - 3F^3_n + F^3_{n-2} = 3F_{3n}$.