ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-34 *Proposed by Paul F. Byrd, San Jose State College, San Jose, California*

Derive the series expansions

$$J_{2k}(\alpha) = I_k^2(\alpha) + \sum_{m=1}^{\infty} (-1)^{m+k} I_{m+k}(\alpha) I_{m-k}(\alpha) L_{2m}$$

($k = 0, 1, 2, 3, \ldots$) for the Bessel functions J_{2k} of all even orders, where L_n are Lucas numbers and I_n are modified Bessel functions.

H-35 *Proposed by Walter W. Horner, Pittsburgh, Pa.*

Select any nine consecutive terms of the Fibonacci sequence and form the magic square

u_8	u_1	u_6
u_3	u_5	u_7
u_4	u_9	u_2

show

$$u_8u_1u_6 + u_3u_5u_7 + u_4u_9u_2 =$$

$$u_8u_3u_4 + u_1u_5u_9 + u_6u_7u_2 .$$

Generalize.

H-36 *Proposed by J.D.E. Konhauser, State College, Pa.*

Consider a rectangle R. From the upper right corner of R remove a rectangle S (similar to R and with sides parallel to the sides of R. Determine the linear ratio $K = L_R/L_S$ if the centroid of the remaining L shaped region is where the lower left corner of the removed rectangle was.

H-37 *Proposed by H.W. Gould, West Virginia University, Morgantown, West. Va.*

Find a triangle with sides n+1, n, n-1 having integral area. The first two examples appear to be 3, 4, 5 with area 6; and 13, 14, 15 with area 84.

H-38 *Proposed by R.G. Buschman, Suny, Buffalo, N.Y.*

(See Fibonacci Numbers, Chebyshev Polynomials, Generalizations and Difference Equations Vol. 1, No. 4, Dec. 1963, pp. 1-7.)

Show

$$(u_{n+r} + (-b)^r u_{n-r})/u_n = \lambda_r$$

H-39 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Solve the difference equation in closed form

$$C_{n+2} = C_{n+1} + C_n + F_{n+2} \;,$$

where $C_1 = 1$, $C_2 = 2$, and F_n is the nth Fibonacci number. Give two separate characterizations of these numbers.

H-40 *Proposed by Walter Blumberg, New Hyde Park, L.I., N.Y.*

Let U, V, A and B be integers, subject to the following conditions

$$\text{(i)}\;\; U > 1, \qquad \text{(ii)}\;\; (U, 3) = 1; \qquad \text{(iii)}\;\; (A, V) = 1;$$

$$\text{(iv)}\;\; V = \sqrt{(u^2-1)/5}\;.$$

Show A^2U+BV is not a square.

SOLUTIONS

EXPANSIONS OF BESSEL FUNCTIONS IN TERMS OF FIBONACCI NUMBERS

P-2 *Proposed by P.F. Byrd, San Jose State College, San Jose, California.*

Derive the series expansions

$$J_0(\alpha) = \sum_{k=0}^{\infty} (-1)^k \quad I_k^2(\alpha) - I_{k+1}^2(\alpha) \quad F_{2k+1} ,$$

where J_0 and I_k are Bessel functions, with F_{2k+1} being Fibonacci numbers.

Solution by the proposer, P.F. Byrd, San Jose State College, San Jose, California

Note: This is a corrected version. Initially this read $J_0(x) = \ldots$ which does not make sense.

It is just as easy to derive the more general series expansions of $J_{2p}(\alpha)$, (p=0, 1, 2, . . .), for Bessel functions of all even orders, and then to obtain the desired result as a special case upon setting p = 0. We make principal use of formulas (6. 1) and (6. 5) presented in [1]. Since $J_{2p}(\alpha)$ is an even function, we first seek a polynomial expansion of the form

$$J_{2p}(\alpha) = \sum_{k=0}^{\infty} \beta_{2k,\,p}(\alpha)\ \varphi_{2k+1}(x) ,$$

where from equation (6. 5) and [1] the coefficients are given by

$$\beta_{2k,\,p}(\alpha) = \frac{(i)^{2k}}{\pi} \int_0^{\pi} J_{2p}(-2i\alpha\cos v)\left[\cos 2kv - \cos(2k+2)v\right]\ dv ,$$

with $_{2k+1}$ being the Fibonacci polynomials defined in [1]. Now it is known (e.g., see [2]) that

$$\int_0^{\pi} J_{2p}(-2i\alpha \cos v) \cos 2mv\ dv = \pi\ J_{p+m}(-i\alpha)\ J_{p-m}(-i\alpha) ,$$

and also that $J_n(iz) = i^n I_n(-z) = (-i)^n I_n(z)$. Hence we easily obtain

$$\beta_{2k,\,p} = (-1)^{p+k}\left[I_{k+p}(\alpha)I_{k-p}(\alpha) - I_{k+p+1}(\alpha)I_{k-p+1}(\alpha)\right] .$$

Finally, taking $x = 1/2$, and thus with $\varphi_{2k+1}(1/2) = F_{2k+1}$, we have the formal expansions

$$J_{2p}(\alpha) = \sum_{k=0}^{\infty} (-1)^{k+p} \left[I_{k+p}(\alpha) I_{k-p}(\alpha) - I_{k+p+1}(\alpha) I_{k-p+1}(\alpha) \right] F_{2k+1} \; .$$

which in particular yield the solution to the proposed problem upon setting $p = 0$.

REFERENCES

1. P. F. Byrd, Expansion of Analytic Functions in Polynomials Associated with Fibonacci Numbers, The Fibonacci Quarterly, Vol. 1, No. 1, pp. 16-29.
2. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge, 2nd Edition, 1944, p. 151.

Note: Corrected statements to P-1.

Verify that the polynomials $\varphi_{k+1}(x)$ satisfy the differential equation

$$(1+x^2)y'' + 3xy' - k(k+2)y = 0 \; ,$$

$(k = 0, 1, 2, 3, \ldots)$.

Readers are requested to submit solutions to the problems in the above mentioned reference [1].

CORRECTIONS IN SAME PAPER

Page 19 Replace 2] by [2] in line 11.

Page 20 In line 5 read (3) as relation[3] referring to footnote 3 In view of . . . .

Page 21 Read $\varphi_{j+1(x)}$ as $\varphi_{j+1}(x)$ in (4.7)

Page 23 In (5.6) place absolute value bars around the quantity approaching the limit zero.

SYMBOLIC RELATIONS

H-18 *Proposed by R.G. Buschman, Oregon State University, Corvallis, Oregon*

"Symbolic relations" are sometimes used to express identities. For example, if F_n and L_n denote, respectively, Fibonacci and

Lucas numbers, then

$$(1 + L)^n \doteq L_{2n}, \qquad (1 + F)^n \doteq F_{2n}$$

are known identities, where $\doteq$ denotes that the exponents on the symbols are to be lowered to subscripts after the expansion is made.

(a) Prove $(L + F)^n \doteq (2F)^n$.

(b) Evaluate $(L + L)^n$.

(c) Evaluate $(F + F)^n$.

(d) How can this be suitably generalized?

Solution by the proposer (now at Suny, Buffalo, N.Y.)

Consider the generating functions:

$$2\,e^{u/2}\sinh(u\sqrt{5}/2) = \sqrt{5}\sum_{n=0}^{\infty} F_n\, u^n/n! \quad ,$$

$$2\,e^{u/2}\cosh(u\sqrt{5}/2) = \sum_{n=0}^{\infty} L_n\, u^n/n! \quad .$$

From these and the product formula for power series we can write

$$\text{(a)}\quad \sqrt{5}\sum_{n=0}^{\infty} F_n\, 2^n u^n/n! = 2\,e^u \sinh(u\sqrt{5}) = \sqrt{5}\sum_{n=0}^{\infty}\sum_{k=0} \frac{F_k\, L_{n-k}}{k!\,(n-k)!}\, u^n \quad ,$$

$$\text{(b)}\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{L_k\, L_{n-1}}{k!\,(n-k)!}\, u^n = 2\,e^u(\cosh u\sqrt{5} + 1) = \sum_{n=0}^{\infty} \frac{L_n\, 2^n + 2}{n!}\, u^n \quad ,$$

$$\text{(c)}\quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{F_k\, F_{n-k}}{k!\,(n-k)!}\, u^n = 2\,e^u(\cosh u\sqrt{5} - 1)/\sqrt{5} = \sum_{n=0}^{\infty} \frac{L_n\, 2^n - 2}{n!}\, u^n \quad .$$

Equating coefficients and multiplying by $n!$ then gives

$$\text{(a)}\quad \sum_{k=0}^{n}\binom{n}{k} F_k\, L_{n-k} = 2^n F_n \qquad \text{or} \quad (F + L)^n = (2F)^n \, ,$$

$$\text{(b)}\quad \sum_{k=0}^{n}\binom{n}{k} L_k\, L_{n-k} = 2^n L_n + 2 \qquad \text{or} \quad (L + L)^n = (2L)^n + 2 \, ,$$

$$\text{(c)}\quad \sum_{k=0}^{n}\binom{n}{k} F_k\, F_{n-k} = (2^n L_n - 2)/5 \qquad \text{or} \quad (F + F)^n = (2L)^n - 2)/5 \, .$$

Solution by L. Carlitz, Duke University, Durham, N.C.

As noted by Gould (this Quarterly, Vol. 1 (1963), p. 2) we have

$$\frac{e^{ax} - e^{bx}}{a-b} = \sum_{n=0}^{\infty} \frac{x^n}{n!} F_n , \qquad e^{ax} + e^{bx} = \sum_{n=0}^{\infty} \frac{x^n}{n!} L_n ,$$

$$a = \frac{1}{2}(1 + \sqrt{5}), \quad b = \frac{1}{2}(1 - \sqrt{5}) .$$

It follows at once that

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} (L + F)^n = \frac{e^{2ax} - e^{2bx}}{a-b} ,$$

so that

(a) $$(L + F)^n = 2^n F_n .$$

Similarly

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} (L + L)^n = e^{2ax} + 2e^{(a+b)x} + e^{2bx}$$
$$= e^{2ax} + e^{2bx} + 2e^x ,$$

so that

(b) $$(L + L)^n = 2^n L_n + 2 ,$$

while

$$(\alpha - \beta)^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} (F + F)^n = e^{2ax} + e^{2bx} - 2e^x ,$$

so that

(c) $$5(F + F)^n = 2^n L_n - 2 .$$

To generalize these formulas consider

$$(e^{ax} + e^{bx})^r = \sum_{s=0}^{r} \binom{r}{s} e^{(r-s)ax+sbx} =$$

$$= \frac{1}{2} \sum_{s=0}^{r} \binom{r}{s} e^{sx} (e^{(r-2s)ax} + e^{(r-2s)bx})$$

$$= \frac{1}{2} \sum_{s=0}^{r} \binom{r}{s} e^{sx} \sum_{n=0}^{\infty} \frac{(r-2s)^n x^n}{n!} L_n .$$

Therefore

$$(L+L+\ldots+L)^n = \frac{1}{2}\sum_{s=0}^{r}\binom{r}{s}\sum_{k=0}^{n}\binom{n}{k}(r-2s)^k s^{n-k} L_k$$

$$= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} L_k \sum_{s=0}^{r}\binom{r}{s}(r-2s)^k s^{n-k} .$$

In particular

$$(L+L+L)^n = 3^n L_n + 3L_{2n} .$$

Similarly, since

$$5^r(F+\ldots+F)^n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} L_k \sum_{s=0}^{2r}(-1)^s\binom{2r}{s}(2r-2s)^k s^{n-k} ,$$

where the number of F's is 2r, and

$$5^r(F+\ldots+F)^n = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} F_k \sum_{s=0}^{2r-1}(-1)^s\binom{2r+1}{s}(2r-2s+1)^k s^{n-k} ,$$

where the number of F's is 2r+1. In particular

$$5(F+F+F)^n = 3^n F_n + 3F_{2n} .$$

A formula for

$$(L + \ldots + L + F + \ldots + F)^n$$

can be obtained but it is very complicated. When the number of L's is equal to the number of F's it is less complicated. In particular

$$(L+L+F+F)^n = \frac{1}{5}(4^n L_n - 2^{n+1}) .$$

Indeed since

$$(e^{ax} + e^{bx})^r(e^{ax} - e^{bx})^r = (e^{2ax} - e^{2bx})^r$$

it follows that

$$(\underbrace{L + \ldots + L}_{r} + \underbrace{F + \ldots + F}_{r})^n = 2^n(\underbrace{F + \ldots + F}_{r})^n .$$

In particular

$$(L + L + L + F + F + F)^n = \frac{2^n}{5}(3\ F_n - 3F_{2n}) .$$

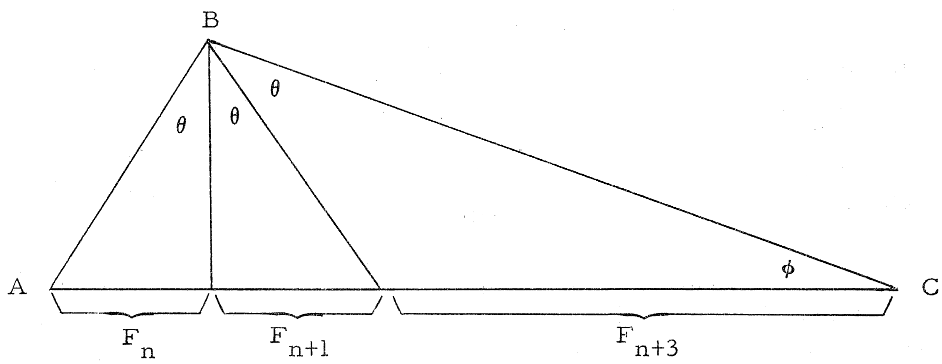
THE RACE

H-19 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas*

In the triangle below [drawn for the case (1, 1, 3)], the trisectors of angle, B, divide side, AC, into segments of length F_n, F_{n+1}, F_{n+3}. Find:

(i) $\lim_{n \to \infty} \theta$

(ii) $\lim_{n \to \infty} \varphi$



Solution by Michael Goldberg, Washington, D.C.

As $n \to \infty$, the ratio F_{n+1}/F_n approaches $t = (\sqrt{5} + 1)/2$, and F_{n+3}/F_n approaches $t^3 = 2t + 1$. Hence, the limiting triangle ABC can be drawn by taking points D and E on AC so that AD = 1, DE = t and EC = $t^3 = 2t + 1$. Since BD is a bisector of angle ABE, the point B must lie on the circle which is the locus of points whose distances to A and E are in the ratio AD/DE = 1/t. The circle passes through D. If the diameter of the circle is $2r = x + 1$, then $x/(x + 1 + t) = 1/t$ from which

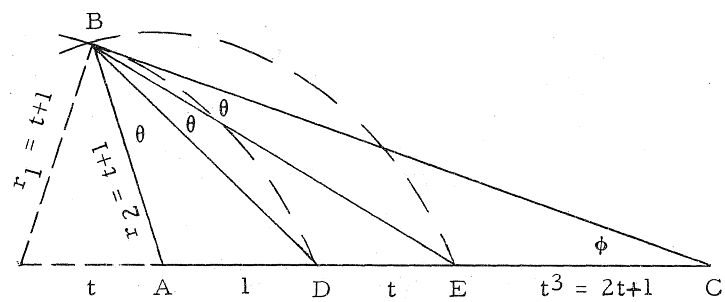
$$r_1 = t/(t - 1) = t^2 = t + 1 \quad .$$

Similarly, BE is a bisector of the angle DBC. The point B must lie on a circle which is the locus of points whose distances from D and C are in the ratio $DE/EC = t/t^3 = 1/t^2$. If the diameter of the circle is $2r_2 = y + t$, then $y/(y + t + t^2) = 1/t^2$ from which

$$r_2 = t_2^2 = t + 1 = r_1 \ .$$

Hence, $\cos \angle BAE = -t/2(t+1) = -(\sqrt{5}-1)/4$ and $\angle BAE = 108°$. From which $2\theta = 90° - 108°/2 = 36°$; $\theta = 18°$, $\phi = 180° - 108° - 3\theta = 18°$

Also solved by the proposer and Raymond Whitney, Penn. State University, Hazelton, Penn.



FIBONACCI TO LUCAS

H-20 *Proposed by Verner E. Hoggatt, Jr., and Charles H. King, San Jose State College, San Jose, California.*

If

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{show} \quad D(e^{Q^n}) = e^{L_n},$$

where $D(A)$ is the determinant of matrix A and L_n is the n^{th} Lucas number.

Solution by John L. Brown, Jr., Penn. State University, State College, Penn.

Recall that

$$Q_n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

so that (by definition)

$$e^{Q^n} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} & \sum_{k=0}^{\infty} \frac{F_{nk}}{k!} \\ \sum_{k=0}^{\infty} \frac{F_{nk}}{k!} & \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} \end{pmatrix}$$

It is well-known that

$$\sum_{k=0}^{\infty} \frac{F_{nk}}{k!} x^k = \frac{e^{a^n x} - e^{b^n x}}{\sqrt{5}}$$

[e.g. equation (2.11), p. 5 of Gould's paper in Vol. 1, No. 2, April 1963], where

$$a = \frac{1+\sqrt{5}}{2}$$

and

$$b = \frac{1-\sqrt{5}}{2} .$$

Similarly,

$$\sum_{k=0}^{\infty} \frac{L_{nk}}{k!} x^k = e^{a^n x} + e^{b^n x}$$

for the Lucas numbers.

But $L_{nk} = F_{nk+1} + F_{nk-1}$; therefore,

$$\sum_{k=0}^{\infty} \frac{F_{nk+1} + F_{nk-1}}{k!} = e^{a^n} + e^{b^n}, \qquad \text{or}$$

(1) $$\sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} = (e^{a^n} + e^{b^n}) - \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} .$$

Since $F_{nk+1} = F_{nk} + F_{nk-1}$ and $\displaystyle\sum_{k=0}^{\infty} \frac{F_{nk}}{k!} = \frac{e^{a^n} - e^{b^n}}{\sqrt{5}}$

from above, we also have

(2) $$\sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} = \sum_{k=0}^{\infty} \frac{F_{nk}}{k!} + \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} = \frac{e^{a^n} - e^{b^n}}{\sqrt{5}} + \sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} .$$

Solving (1) and (2) simultaneously, we find

(3) $$\sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!} = \frac{1}{2}\left[\left(e^{a^n} + e^{b^n}\right) + \frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right]$$

and

(4) $$\sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!} = \frac{1}{2}\left[\left(e^{a^n} + e^{b^n}\right) - \left(\frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right)\right]$$

Now, $$D(e^{Q^n}) = \left(\sum_{k=0}^{\infty} \frac{F_{nk+1}}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{F_{nk-1}}{k!}\right) - \left(\sum_{k=0}^{\infty} \frac{F_{nk}}{k!}\right)^2$$

$$= \frac{1}{4}\left[\left(e^{a^n} + e^{b^n}\right)^2 - \left(\frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right)^2\right] - \left(\frac{e^{a^n} - e^{b^n}}{\sqrt{5}}\right)^2$$

$$= e^{a^n + b^n} = e^{L_n}, \text{ since } L_n = a^n + b^n \text{ for } n \geq 0. \quad \underline{\text{q. e. d.}}$$

FIBONACCI PROBABILITY

H-21 *Proposed by Francis D. Parker, University of Alaska, College, Alaska*

Find the probability, as n approaches infinity, that the n^{th} Fibonacci number, $F(n)$, is divisible by another Fibonacci number ($\neq F_1$ or F_2).

Solution by proposer

We use frequently the fact that $F(n)$ is divisible by $F(k)$ if k divides n. Then the probability that $F(n)$ is divisible by 2 is $1/3$; the probability that $F(n)$ is divisible by 3 but <u>not</u> 2 is $(\frac{1}{4})(\frac{2}{3})$; the probability that $F(n)$ is divisible by 5 but not by 2 or by 3 is $\frac{1}{5}\,\frac{2}{3}\,\frac{3}{4} = \frac{2}{4.5}$; and in general the probability that $F(n)$ is divisible by $F(k)$ but not any Fibonacci number of order less than k is $\frac{2}{k(k-1)}$. These probabilities are all independent, so that the probabilities that $F(n)$ is divisible by at least one Fibonacci number of order not exceeding k is

$$\frac{3}{3.2} + \frac{2}{4.3} + \frac{2}{5.4} + \cdots \frac{2}{k(k-1)} \quad .$$

This sum is $\frac{k-2}{k}$, and as n approaches infinity, the probability approaches unity.

Also solved by J. L. Brown, Jr., Penn. State University, State College, Pennsylvania.