SOME DETERMINANTS INVOLVING POWERS OF FIBONACCI NUMBERS

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In the October, 1963 issue of this journal [1], the author discussed some of the periodic properties of Fibonacci summations. It was noted that a certain determinant was basic to these considerations. Its main characteristics and value were indicated and a promise was given of additional explanation in some later issue of the Fibonacci Quarterly. The purpose of this article is to set forth the manner of evaluating these determinants on an empirical basis. The proof of the general validity of the results obtained is to be found in an article by Terry Brennan in this issue of the Quarterly [2].

To fix ideas the determinant of the sixth order will be used. Written in a form that brings out its Fibonacci characteristics it would be:

$$\begin{vmatrix} 1^6+1^6-1 & 1^5 1 & 1^4 1^2 & 1^3 1^3 & 1^2 1^4 & 1\,1^5 \\ 2^6+1^6-1 & 2^5 1 & 2^4 1^2 & 2^3 1^3 & 2^2 1^4 & 2\,1^5 \\ 3^6+2^6-1 & 3^5 2 & 3^4 2^2 & 3^3 2^3 & 3^2 2^4 & 3\,2^5 \\ 5^6+3^6-1 & 5^5 3 & 5^4 3^2 & 5^3 3^3 & 5^2 3^4 & 5\,3^5 \\ 8^6+5^6-1 & 8^5 5 & 8^4 5^2 & 8^3 5^3 & 8^2 5^4 & 8\,5^5 \\ 13^6+8^6-1 & 13^5 8 & 13^4 8^2 & 13^3 8^3 & 13^2 8^4 & 13\,8^5 \end{vmatrix} .$$

A certain subtlety should be noted in the first line as the first "1" stands for F_2 and the second for F_1.

By separating the terms of the first column into groups, the problem can be changed to that of evaluating three determinants with first columns as indicated below:

(1)	(2)	(3)
1	1	-1
2^6	1	-1
3^6	2^6	-1
5^6	3^6	-1
8^6	5^6	-1
13^6	8^6	-1

Determinants (1) and (2) can be evaluated in terms of what shall be called the BASIC POWER DETERMINANT. Determinant (3) will be developed in terms of the cofactors of the first column which involve the basic power determinant minus one of its rows.

BASIC POWER DETERMINANT

The first determinant has a common factor in each of its rows (the factors are 1, 2, 3, 5, 8, 13 respectively). If these factors be taken out of the determinant, we have what will be called the basic power determinant. For the sixth order, it is as shown below:

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2^5 & 2^4 & 2^3 & 2^2 & 2 & 1 \\ 3^5 & 3^4 2 & 3^3 2^2 & 3^2 2^3 & 3 \cdot 2^4 & 2^5 \\ 5^5 & 5^4 3 & 5^3 3^2 & 5^2 3^3 & 5 \cdot 3^4 & 3^5 \\ 8^5 & 8^4 5 & 8^3 5^2 & 8^2 5^3 & 8 \cdot 5^4 & 5^5 \\ 13^5 & 13^4 8 & 13^3 8^2 & 13^2 8^3 & 13 \cdot 8^4 & 8^5 \end{vmatrix} .$$

This determinant is a special case of the more general determinant in which the first row starts with any Fibonacci number whatsoever

$$\begin{vmatrix} F_i^5 & F_i^4F_{i-1} & F_i^3F_{i-1}^2 & F_i^2F_{i-1}^3 & F_iF_{i-1}^4 & F_{i-1}^5 \\ F_{i+1}^5 & F_{i+1}^4F_i & F_{i+1}^3F_i^2 & F_{i+1}^2F_i^3 & F_{i+1}F_i^4 & F_i^5 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ F_{i+5}^5 & F_{i+5}^4F_{i+4} & F_{i+5}^3F_{i+4}^2 & F_{i+5}^2F_{i+4}^3 & F_{i+5}F_{i+4}^4 & F_{i+4}^5 \end{vmatrix} .$$

The basis for evaluating this determinant is the relation

$$F_nF_{n+k+1} - F_{n+1}F_{n+k} = (-1)^{n+1}F_k .$$

To evaluate the determinant we proceed to produce zeros in the first row. This is done by multiplying the first column by F_{i-1} and subtracting from this F_i times the second column; then multiplying the second column by F_{i-1} and subtracting from this F_i times the third column; etc. The operations for the first and second columns would be as follows:

$$F_{i+1}^4(F_{i-1}F_{i+1} - F_iF_i) = (-1)^iF_1F_{i+1}^4 ,$$

$$F_{i+2}^4(F_{i-1}F_{i+2} - F_iF_{i+1}) = (-1)^iF_2F_{i+2}^4 ,$$

$$F_{i+3}^4(F_{i-1}F_{i+3} - F_iF_{i+2}) = (-1)^iF_3F^4_{i+3} ,$$

and so on. It is clear that the second row would have a factor F_1, the third a factor F_2, etc. Thus, after eliminating the common factors and expanding by the non-zero term in the last column of the row, the absolute value of the resulting determinant would be $F_1F_2 \cdot F_3F_4F_5$ multiplied by the basic power determinant of the fifth order. If we adopt the notation $\triangle n$

to represent the basic power determinant of the nth order, this result may be expressed as

$$|\Delta_6| = \prod_{i=1}^{5} (F_i) \cdot |\Delta_5| .$$

In general, for a determinant of order n,

$$|\Delta_n| = \prod_{i=1}^{n-1} (F_i) \cdot |\Delta_{n-1}| .$$

Since the process may be repeated, it is not difficult to arrive at the final result:

$$|\Delta_n| = \prod_{i=1}^{n-1} F_i^{n-i} .$$

In the particular case of order six,

$$|\Delta_6| = F_1^5 F_2^4 F_3^3 F_4^2 F_5 = 2^3 3^2 \cdot 5 .$$

It is interesting to note that the values of these basic power determinants are independent of where we start in the Fibonacci sequence.

SIGN OF THE BASIC POWER DETERMINANT

It is important to be able to determine the sign of the basic determinant value inasmuch as we shall combine the values of determinants (1) and (2) with the values of the cofactors of determinant (3). The considerations involved are a bit tedious. We distinguish four cases according as n is of the form 4k, 4k+1, 4k+2, or 4k+3. The following three factors determine the outcome:

(i) The sign introduced by expanding from the last element in the first row;

(ii) The signs of the terms of the determinant resulting after each of the steps indicated above. These terms will be either all plus or all minus.

(iii) The sign of Δ_2 which is the second-order determinant of the first powers of the Fibonacci numbers in the last two rows. Thus, for the sixth order determinant we have been considering, Δ_2 is

$$\begin{vmatrix} 8 & 5 \\ 13 & 8 \end{vmatrix} = -1 \quad .$$

The final outcome is as follows:

(i) For order 4k or 4k+1, the sign is always plus;

(ii) For order 4k+2 or 4k+3, the sign agrees with that of Δ_2.

As noted previously, the basic power determinant enables us to evaluate determinants (1) and (2). The latter can be brought to this form by shifting the first column so that it becomes the last column.

BASIC POWER DETERMINANTS WITH ONE ROW MISSING

To evaluate the third determinant we find the cofactors of the elements in the first column. For the element in the first row, this cofactor is a basic power determinant after removing common factors, but for all the others it is essentially a basic power determinant with one row missing. The absolute value of such a determinant of order n with a missing row between the k and (k+1)st row will be represented by

$$A_n(k \mid k+1)$$

the implication being that the absolute value does not depend on the particular Fibonacci number with which it starts. When developing such a determinant the procedure is the same as for the development of the basic power determinant, only in this case there is a gap. The calculation for a determinant of order n with a row missing between the third and fourth rows can be summarized schematically in the following manner. The column headings are Fibonacci numbers. A table entry is the power to which the Fibonacci number at the head of the column is being raised. The quantities in any one row are multiplied together. In the first row we have the result of the first step in the evaluation in which the order is changed from 9 to 8; in the second, the factors resulting in reducing the determinant from order 8 to order 7; etc.

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9
1	1	0	1	1	1	1	1	1
1	0	1	1	1	1	1	1	
0	1	1	1	1	1	1		
1	1	1	1	1	0			
1	1	1	1	0				
1	1	1	0					
1	1	0						
1	0							
0								

The sum of the quantities in any column gives the power of the Fibonacci number in the determinant. In the above case

$$A_9(3 \mid 4) = F_1^7 F_2^6 F_3^5 F_4^5 F_5^4 F_6^3 F_7^3 F_8^2 F_9 \ .$$

The same result would have been obtained if the gap had been after the sixth row. In general, if the determinant is of order n, a gap after the kth row or the (n-k)th row gives the same result.

The pattern observed is as follows: (1) A reduction of 2 in the powers of F_1 to F_k inclusive (if k is less than n-k); (2) A reduction of 1 from F_{k+1} to F_{n-k} inclusive; (3) No reduction thereafter. If n-k is less than k, the roles of k and n-k are reversed. Finally, if n-k equals k (even n), there would be a reduction of 2 from 1 to k and no reduction thereafter.

These results may be summarized in the following formulas.

FORMULA FOR k LESS THAN n-k

$$A_n(k \mid k+1) = \prod_{i=1}^{k} F_i^{n-i-1} \prod_{i=k+1}^{n-k} F_i^{n-i} \prod_{i=n-k+1}^{n} F_i^{n-i+1} \ ,$$

FORMULA FOR n-k LESS THAN k

$$A_n(k \mid k+1) = \prod_{i=1}^{n-k} F_i^{n-i-1} \prod_{i=n-k+1}^{k} F_i^{n-i} \prod_{i=k+1}^{n} F_i^{n-i+1} \ ,$$

FORMULA FOR k EQUAL TO n-k

$$A_n(n/2 \mid n/2+1) = \prod_{i=1}^{n/2} F_i^{n-i-1} \prod_{i=\frac{n}{2}+1}^{n} F_i^{n-i+1} .$$

These formulas are not difficult of application. However, for the sake of convenience (in view of future considerations) and as a possible guide to readers the results for orders 12 and 13 are set down in detail. Since, however, there is symmetry in k and n-k only the first half need be given in each case.

TABLE OF $A_{12}(k \mid k+1)$

k	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	10	10	9	8	7	6	5	4	3	2	1	1
2	10	9	9	8	7	6	5	4	3	2	2	1
3	10	9	8	8	7	6	5	4	3	3	2	1
4	10	9	8	7	7	6	5	4	4	3	2	1
5	10	9	8	7	6	6	5	5	4	3	2	1
6	10	9	8	7	6	5	6	5	4	3	2	1

TABLE OF $A_{13}(k \mid k+1)$

k	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
1	11	11	10	9	8	7	6	5	4	3	2	1	1
2	11	10	10	9	8	7	6	5	4	3	2	2	1
3	11	10	9	9	8	7	6	5	4	3	3	2	1
4	11	10	9	8	8	7	6	5	4	4	3	2	1
5	11	10	9	8	7	7	6	5	5	4	3	2	1
6	11	10	9	8	7	6	6	6	5	4	3	2	1

SIGN OF POWER DETERMINANT WITH ONE LINE MISSING

The considerations leading to the determination of the sign of power determinants with a line missing are involved. The approach is precisely the same as for the power determinant. The results for all values of n, i (subscript of the leading Fibonacci number in the determinant) and k (as defined for the break point but taken modulo 4) are listed in the following table.

k	4r	4r+1	4r+2 i odd	4r+2 i even	4r+3 i odd	4r+3 i even
1	-	+	+	-	+	-
2	-	-	+	-	-	+
3	+	-	-	+	-	+
4	+	+	-	+	+	-

EVALUATION OF THE ORIGINAL DETERMINANT

We noted previously that the original determinant could be represented as the sum of three separate determinants (1), (2), and (3). Determinant (1) is simply the product of Fibonacci numbers (one from each row) by the basic power determinant. Thus for the sixth order, the situation would be as follows:

	F_1	F_2	F_3	F_4	F_5	F_6	F_7
BPD	5	4	3	2	1		
F's		1	1	1	1	1	1
(1)	5	5	4	3	2	1	1

The sign would be negative.

Determinant (2) can be related to the basic power determinant by moving the first column into the last position. For the sixth order, this involves a change of sign. Again factors can be taken out leaving a basic power determinant. The pattern is as follows:

	F_1	F_2	F_3	F_4	F_5	F_6
BPD	5	4	3	2	1	
F's	1	1	1	1	1	1
(2)	6	5	4	3	2	1

The sign would be positive.

To evaluate (3) we expand by the first column. We shall designate successive elements of the expansion, due account being taken of all signs including the negative quantities in the first column, by successive capital letters: A, B, C, D, ... A gives rise to a simple basic power determinant; B to one with a line missing (k = 1); C with the second line missing (k = 2); etc. However, there are factors that have to be multiplied in each case. It should be noted too that we are referring to determinants of the fifth order and not of the sixth.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	4	3	2	1				
			2	2	2	2	1	
A	4	3	4	3	2	2	1	(negative)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	3	2	1	1			
			1	2	2	2	1	
B	3	3	3	3	3	2	1	(positive)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	2	2	2	1			
			1	1	2	2	1	
C	3	2	3	3	3	2	1	(positive)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	2	2	2	1			
			2	1	1	2	1	
D	3	2	4	3	2	2	1	(negative)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	3	3	2	1	1			
			2	2	1	1	1	
E	3	3	4	3	2	1	1	(negative)

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	
	4	3	2	1				
			2	2	2	1		
F	4	3	4	3	2	1		(positive)

Summarizing in one table (omitting the first and second Fibonacci number factors as they are both unity) we have the following for the evaluation of the determinant of the sixth order.

	Sign	F_3	F_4	F_5	F_6	F_7
(1)	-	4	3	2	1	1
(2)	+	4	3	2	1	
A	-	4	3	2	2	1
B	+	3	3	3	2	1
C	+	3	3	3	2	1
D	-	4	3	2	2	1
E	-	4	3	2	1	1
F	+	4	3	2	1	

The following pairs of terms combine: E and (1); F and (2); A and D; B and C. The resulting sums have a common factor of $2^8\ 3^3\ 5^2$, the adjoint factor being 144. Thus finally the value of the sixth order determinant is found to be $2^{12}\ 3^5\ 5^2$.

DETERMINANT OF ORDER 12

Without justifying all the intermediate steps, the summation table for order 12 is shown below.

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
(1)	+	10	9	8	7	6	5	4	3	2	1	1
(2)	-	10	9	8	7	6	5	4	3	2	1	
A	+	10	9	8	7	6	5	4	3	2	2	1
B	-	9	9	8	7	6	5	4	3	3	2	1
C	-	9	8	8	7	6	5	4	4	3	2	1
D	+	9	8	7	7	6	5	5	4	3	2	1

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
E	+	9	8	7	6	6	6	5	4	3	2	1
F	-	9	8	7	6	6	6	5	4	3	2	1
G	-	9	8	7	7	6	5	5	4	3	2	1
H	+	9	8	8	7	6	5	4	4	3	2	1
I	+	9	9	8	7	6	5	4	3	3	2	1
J	-	10	9	8	7	6	5	4	3	2	2	1
K	-	10	9	8	7	6	5	4	3	2	1	1
L	+	10	9	8	7	6	5	4	3	2	1	

It will be noted that the following pairs add up to zero: E and F; D and G; C and H; B and I; A and J; (1) and K; (2) and L. Therefore, the value of the determinant is zero. The same result was found for orders 4, 8, and 16.

DETERMINANT OF ORDER 13

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
(1)	+	11	10	9	8	7	6	5	4	3	2	1	1
(2)	+	11	10	9	8	7	6	5	4	3	2	1	
A	-	11	10	9	8	7	6	5	4	3	2	2	1
B	-	10	10	9	8	7	6	5	4	3	3	2	1
C	+	10	9	9	8	7	6	5	4	4	3	2	1
D	+	10	9	8	8	7	6	5	5	4	3	2	1
E	-	10	9	8	7	7	6	6	5	4	3	2	1
F	-	10	9	8	7	6	7	6	5	4	3	2	1
G	+	10	9	8	7	7	6	6	5	4	3	2	1
H	+	10	9	8	8	7	6	5	5	4	3	2	1
I	-	10	9	9	8	7	6	5	4	4	3	2	1
J	-	10	10	9	8	7	6	5	4	3	3	2	1
K	+	11	10	9	8	7	6	5	4	3	2	2	1
L	+	11	10	9	8	7	6	5	4	3	2	1	1
M	-	11	10	9	8	7	6	5	4	3	2	1	

It will be noted that the following pairs add up to zero: E and G; C and I; A and K; (2) and M. The others can be combined to give the following table.

	Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
(1), L	+	12	10	9	8	7	6	5	4	3	2	1	1
B, J	-	11	10	9	8	7	6	5	4	3	3	2	1
D, H	+	11	9	8	8	7	6	5	5	4	3	2	1
F	-	10	9	8	7	6	7	6	5	4	3	2	1

After taking out the common factor

$$F_3^{10}F_4^9F_5^8F_6^7F_7^6F_8^6F_9^5F_{10}^4F_{11}^3F_{12}^2F_{13}F_{14}$$

the following remains for evaluation:

Sign	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
+	2	1	1	1	1							
-	1	1	1	1	1					1	1	
+	1			1	1			1	1	1	1	
-						1	1	1	1	1	1	

No easy method was found for evaluating the sum of these quantities. Essentially it was a matter of evaluating them, combining them and then factoring. Fortunately, as the numbers to be factored increased in size going up to 23 digits in one instance, a pattern involving Fibonacci and Lucas numbers was discovered with the result that the formulas (1) and (2) on page 38 of [1] were discovered.

The matter can be allowed to rest here. The path pursued has been illustrated in sufficient detail to allow others to explore these interesting determinants. The formulas obtained as well as the determinant values to the twentieth order are set forth in the paper [1] and need not be repeated.

REFERENCES

1. Brother U. Alfred, "Periodic Properties of Fibonacci Summations, The Fibonacci Quarterly, 1(1963), No. 3, pp. 33-42.
2. T. L. Brennan "Fibonacci Powers and Pascal's Triangle in a Matrix" The Fibonacci Quarterly, 2(1964), No. 2, pp. 93-103.

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