# A NOTE ON WARING'S FORMULA FOR SUMS OF LIKE POWERS OF ROOTS

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Sums of powers $S_k = x_1^k + x_2^k + \ldots + x_n^k$ may be expressed in terms of elementary symmetric functions or in terms of the coefficients of:

$$f(x) = (x-x_1)(x-x_2) \ldots (x-x_n) = x^n + p_1 x^{n-1} + \ldots + p_n$$

by Newton's formulas, usually introduced in a course in the theory of equations, for example, J. V. Uspensky [1].

The relationship between Waring's formula for sums of like powers of the roots of a quadratic and Lucas numbers is quite obvious although perhaps a little too specialized for L. E. Dickson [2] to have pointed this out in his text, First Course in the Theory of Equations.

In order to obtain an explicit expression for $S_k$ where $k = 1, 2, 3 \ldots$, first consider the quadratic

$$x^2 + px + q = 0 \quad . \tag{1}$$

If we denote the roots by $\alpha$ and $\beta$ then (1) may be rewritten as

$$x^2 + px + q = (x-\alpha)(x-\beta) \quad . \tag{2}$$

After making the transformation $x = 1/y$ and multiplying by $y^2$ we obtain,

$$1 + py + qy^2 = (1-\alpha y)(1-\beta y) \quad . \tag{3}$$

Differentiating both sides of (3) with respect to $y$ and dividing both members of the differentiated equation by the corresponding members of (3) we arrive at

$$(4) \qquad \frac{-p-2qy}{1+py+qy^2} = \frac{\alpha}{1-\alpha y} + \frac{\beta}{1-\beta y} \ .$$

Equations,

$$(5) \qquad \frac{\alpha}{1-\alpha y} = \alpha + \alpha^2 y + \ldots + \alpha^k y^{k-1} + \frac{\alpha^{k+1}}{1-\alpha y} y^k$$

and

$$(6) \qquad \frac{\beta}{1-\beta y} = \beta + \beta^2 y + \ldots + \beta^k y^{k-1} + \frac{\beta^{k+1}}{1-\beta y} y^k$$

are both obtained from the geometric series

$$(7) \qquad \frac{1}{1-r} = \sum_{j=0}^{k-1} r^j + \frac{r^k}{1-r} \quad ;$$

for example, let $r = \alpha y$ and multiply by $\alpha$. Addition of (5) and (6) results in,

$$(8) \quad \frac{\alpha}{1-\alpha y} + \frac{\beta}{1-\beta y} = S_1 + S_2 y + \ldots + S_k y^{k-1} + \frac{\alpha^{k+1}(1-\beta y) + \beta^{k+1}(1-\alpha y)}{(1-\alpha y)(1-\beta y)}$$

where
$$S_k = \alpha^k + \beta^k \ .$$

In order to expand the left-hand member of (4) using (7), let $r = -py$ $-qy^2$, then

$$(9) \qquad \frac{1}{1+py+qy^2} = \sum_{j=0}^{k-1} (-1)^j (py+qy^2)^j + \frac{(-p-qy)^k y^k}{1+py-qy^2}$$

Employing the binomial theorem we may write

$$(py+qy^2)^j = \Sigma \frac{(g+h)!}{g!\,h!} (py)^g (qy^2)^h$$

where the summation is taken over all two-part partitions of j, i.e., for all $g \geq 0$ and $h \geq 0$ such that $g + h = j$. Therefore,

$$\frac{-p-2qy}{1+py-qy^2} = (p+2qy)\,\Sigma\,(-1)^{g+h+1}\,\frac{(g+h)!}{g!\,h!}\,p^g q^h y^{g+2h} + \frac{(-p-2qy)(-p-qy)^k y^k}{1+py+qy^2}. \tag{10}$$

Now the left-hand members of (8) and (10) are equal as shown by equation (4); therefore we may equate coefficients of like powers of y. Specifically, equating coefficients of $y^{k-1}$ we arrive at

$$S_k = \Sigma\,(-1)^{i+j}\,\frac{(i+j-1)!}{(i-1)!\,j!}\,p^i q^i + 2\,\Sigma\,(-1)^{i+j} \cdot \frac{(i+j-1)!}{i!\,(j-1)!}\,p^i q^j \tag{11}$$

where we have replaced i for g + 1 and j for h in (10). The summations in (11) now extend over all $i \geq 0$ and $j \geq 0$ such that $i + 2j = k$. Combining both summations in (11) we have,

$$S_k = k\,\Sigma\,(-1)^{i+j}\,\frac{(i+j-1)!}{i!\,j!}\,p^i q^j \tag{12}$$

summed over all $i \geq 0$, $j \geq 0$ such that $i + 2j = k$. Clearly, for $p = q = -1$ we have $S_k = L_k$, the kth Lucas number; and (12) becomes

$$L_k = k \sum_{\substack{i \geq 0 \\ j \geq 0 \\ i+2j=k}} \frac{(i+j-1)!}{i!\,j!}\,. \tag{13}$$

Equation (12) of Waring's formula, published in 1762, which can be extended to include the sum of kth powers of the roots of n th degree polynomials.

The main point to observe is not necessarily the meager result given by (13) but the fact that implicit in the development of Waring's formula lies the generating function (4) for the Lucas numbers.

REFERENCES

1. J. V. Uspensky, Theory of Equations, McGraw-Hill, 1948.
2. L. E. Dickson, First Course in the Theory of Equations, John Wiley, 1922.

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OMISSIONS

H-3 and H-8 were also solved by John L. Brown, Jr., The Pennsylvania State University, State College, Penn.

The solution to H-16 given in the last issue was compounded from solutions given by L. Carlitz and John L. Brown, Jr. The varitypist omitted the credit line.

H-13 was also solved by John H. Halton, University of Colorado at Boulder, Colorado.

H-15 was also solved by L. Carlitz, Duke University, Durham, N. C.