FIBONACCI POWERS AND PASCAL'S TRIANGLE IN A MATRIX – PART I*

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1. INTRODUCTION

The main point of this paper is to display some interesting properties of the $(n+1) \times (n+1)$ matrix P_n defined by imbedding Pascal's triangle in a square matrix:

(1.1)
$$P_n = \begin{bmatrix} \cdots & 0 & 0 & 0 & 1 \\ \cdots & 0 & 0 & 1 & 1 \\ \cdots & 0 & 1 & 2 & 1 \\ \cdots & 1 & 3 & 3 & 1 \\ & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

The matrix P_n was originally constructed by the author in order to evaluate a determinant presented by Brother U. Alfred. The determinant, and its origin, has subsequently been published in [1] and, for the sake of completeness, its evaluation will be presented here.

2. THE PROBLEM AND ITS SOLUTION

THE PROBLEM:

Evaluate the fifth order determinant

(2.1)
$$\begin{vmatrix} 1^5+1^5-1 & 1^4 1 & 1^3 1^2 & 1^2 1^3 & 1 \cdot 1^4 \\ 2^5+1^5-1 & 2^4 1 & 2^3 1^2 & 2^2 1^3 & 2 \cdot 1^4 \\ 3^5+2^5-1 & 3^4 2 & 3^3 2^2 & 3^2 2^3 & 3 \cdot 2^4 \\ 5^5+3^5-1 & 5^4 3 & 5^3 3^2 & 5^2 3^3 & 5 \cdot 3^4 \\ 8^5+5^5-1 & 8^4 5 & 8^3 5^2 & 8^2 5^3 & 8 \cdot 5^4 \end{vmatrix}$$

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and its n-th order generalization. For the n-th order the powers in the first column would be n and the determinant would extend to u_{n+1} and u_n in the last row, where u_n is the n-th Fibonacci number:

$$u_{n+1} = u_n + u_{n-1} \quad \text{with } u_0 = 0,\ u_1 = 1 . \tag{2.2}$$

The determinant (2.1), which we will call D_5 (D_n in general), will be evaluated as an expansion of cofactors of the first column. In order to keep track of terms in the expansion it is convenient to define D_n for an arbitrary sequence $a_0, a_1, a_2, \ldots, a_{n+1}$ by appropriately placing the members of this sequence in the first column of D_n:

$$D_5\{a\} = \begin{vmatrix} a_2 - 1^5 a_1 - 1^5 a_0 & 1^4 1 & \\ a_3 - 2^5 a_1 - 1^5 a_0 & 2^4 1 & \\ a_4 - 3^5 a_1 - 2^5 a_0 & 3^4 2 & \cdots \\ a_5 - 5^5 a_1 - 3^5 a_0 & 5^4 3 & \\ a_6 - 8^5 a_1 - 5^5 a_0 & 8^4 5 & \end{vmatrix} . \tag{2.3}$$

Clearly (2.1) is $D_5\{a\}$ with $a_0 = a_1 = a_2 = \ldots = a_6 = -1$.

For simplicity we will content ourselves with the reduction of the fifth order determinant (2.3) while mentioning the corresponding results for the general case. The reduction rests on the groundwork of Brother U. Alfred.

THE SOLUTION:

Basic to the reduction is the determinant of the following matrix:

$$B_{5,1} = \begin{bmatrix} 1^4 & 1^3 1 & 1^2 1^2 & 1 \cdot 1^3 & 1^4 \\ 2^4 & 2^3 1 & 2^2 1^2 & 2 \cdot 1^3 & 1^4 \\ 3^4 & 3^3 2 & 3^2 2^2 & 3 \cdot 2^3 & 2^4 \\ 5^4 & 5^3 3 & 5^2 3^2 & 5 \cdot 3^3 & 3^4 \\ 8^4 & 8^3 5 & 8^2 5^2 & 8 \cdot 5^3 & 5^4 \end{bmatrix} \tag{2.4}$$

and in general B_{ni} where n is the order of the matrix and i denotes the first row entries as u_{i+1} and u_i.

An interesting property of the determinant of $B_{5,i}$ is that its magnitude is independent of the index, or starting point, i. This fact is evident when we multiply the two matrices

$$B_{5,i}\, Q_4 = B_{5,i+1} \tag{2.5}$$

where

$$Q_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 6 & 3 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.6}$$

is the matrix of (1.1) "transposed" about its counter diagonal. Since the determinant of Q_4 is ± 1 we have

$$|B_{5,i}| = \pm\, |B_{5,i+1}| .$$

More precisely we can develop

$$\begin{gathered} |Q_0| = 1, \quad |Q_1| = -1, \quad |Q_2| = -1, \ \ldots, \\ |Q_{n-1}| = (-1)^{n(n-1)/2} \quad . \end{gathered} \tag{2.7}$$

We can start, then, with $B_{5,0}$ and shift indices on each row to obtain

$$B_{5,0}\, Q_4^i = B_{5,i} \quad . \tag{2.8}$$

But

$$B_{5,0} = \begin{bmatrix} 1^4 & 0 & 0 & 0 & 0 \\ 1^4 & 1^3 1 & 1^2\, 1^2 & 1 \cdot 1^3 & 1^4 \\ 2^4 & 2^3\, 1 & 2^2\, 1^2 & 2 \cdot 1^3 & 1^4 \\ 3^4 & 3^3\, 2 & 3^2\, 2^2 & 3 \cdot 2^3 & 2^4 \\ 5^4 & 5^3\, 3 & 5^2\, 3^2 & 5 \cdot 3^3 & 3^4 \end{bmatrix}$$

Passing to determinants we have

$$|B_{5,i}| = |Q_4|^i \; |B_{5,0}| = (-1)^{10i}\, 1 \cdot 2 \cdot 3 \cdot 5\, |B_{4,i}|$$

where $|B_{5,0}|$ has been expanded by cofactors of its first row and common row factors have been removed. Having established a recursive process for evaluating $|B_{ni}|$ the general formula may be shown:

$$|B_{ni}| = (-1)^s\, u_{n-1}\, u_{n-2}^2\, u_{n-3}^3 \cdots u_2^{n-2}$$

where $\qquad s = n(n-1)(3i + n-2)/6 .$

In a notation which will be more convenient to use, we define

$$S_o = 1, \quad S_{n+1} = (-1)^n S_n \quad \text{for } n \geq 0$$

$$F_0(x) = 1, \quad F_{n+1}(x) = x_{n+1}\, F_n(x) \quad \text{for any sequence } \{x_n\}$$

$$W_n(x) = F_n(F(x)) .$$

Then

$$(2.9) \qquad |Q_{n-1}| = S_n \quad \text{and} \quad |B_{ni}| = S_n^{i-1} F_n(S) W_{n-1}(u) .$$

Let us see what progress can be made with $|D_n|$. Writing (2.3) as three separate determinants on the first column we have, symbolically,

$$(2.10)\quad D_5\{a\} = \begin{vmatrix} a_2 & \\ a_3 & \\ a_4 & \cdots \\ a_5 & \\ a_6 & \end{vmatrix} - a_1 \begin{vmatrix} 1^5 & \\ 2^5 & \\ 3^5 & \cdots \\ 5^5 & \\ 8^5 & \end{vmatrix} - a_0 \begin{vmatrix} 1^5 & \\ 1^5 & \\ 2^5 & \cdots \\ 3^5 & \\ 5^5 & \end{vmatrix} .$$

The first determinant is the hard one. The second compares nicely with $|B_{5,1}|$ after a common factor is removed from each row. The third becomes $|B_{5,1}|$ when the first column is moved to the last (a change in sign for an even order determinant) and common factors are removed from each row. Hence

$$D_5\{a\} = \begin{vmatrix} a_2 & \\ a_3 & \\ a_4 & \cdots \\ a_5 & \\ a_6 & \end{vmatrix} - a_1\, F_6(u)\, |B_{5,1}| - a_0\, F_5(u)\, |B_{5,1}|$$

and using (2.9)

$$(2.11)\qquad D_5\{a\} = \begin{vmatrix} a_2 & \\ a_3 & \\ a_4 & \cdots \\ a_5 & \\ a_6 & \end{vmatrix} - a_1\, F_5(S)\, F_6(u)\, W_4(u) - a_0\, F_5(u)\, W_5(u) .$$

Expansion of the first determinant by cofactors of its first column gives rise to determinants of the form

$$(2.12)\qquad (1\cdot 2\cdot 5\cdot 8)\cdot(1\cdot 1\cdot 3\cdot 5)a_4 \begin{vmatrix} 1^3 & 1^2\,1 & 1\cdot 1^2 & 1^3 \\ 2^3 & 2^2\,1 & 2\cdot 1^2 & 1^3 \\ 5^3 & 5^2\,3 & 5\cdot 3^2 & 3^3 \\ 8^3 & 8^2\,5 & 8\cdot 5^2 & 5^3 \end{vmatrix}$$

the minor being similar to $|B_{4,1}|$ except that the third row is missing. Our task is to evaluate these minors. Starting with $B_{5,1}$ in (2.4) we form the product

$$(2.13)\qquad B_{5,1}\,Q_4^{-3} = \begin{bmatrix} u_{-1}^4 & u_{-1}^3 u_{-2} & u_{-1}^2 u_{-2}^2 & u_{-1} u_{-2}^3 & u_{-2}^4 \\ 0^4 & 0^3 u_{-1} & 0^2 u_{-1}^2 & 0\cdot u_{-1}^3 & u_{-1}^4 \\ 1^4 & 1^3\,0 & 1^2\,0^2 & 1\cdot 0^3 & 0^4 \\ 1^4 & 1^3\,1 & 1^2\,1^2 & 1\cdot 1^3 & 1^4 \\ 2^4 & 2^3\,1 & 2^2\,1^2 & 2\cdot 1^3 & 1^4 \end{bmatrix}$$

Here, as in (2.8), the matrix Q shifts indices on each row and, overapplying this shift, introduces the negative side of the Fibonacci sequence by way of the relation $u_{n-1} = u_{n+1} - u_n$. Expanding the determinant of (2.13) by the third row we have

$$(2.14)\qquad |B_{5,1}|\;|Q_4|^{-3} = (-1)^2(u_{-2}\,u_{-1})(u_1\,u_2)\,|B_{4,-2}^3|$$

where $B_{4,-2}^3$ is $B_{4,-2}$ (i.e., the fourth order matrix of (2.4) with u_{-1} and u_{-2} in its first row) and where the superscript 3 denotes that the third row is missing.

$$(2.15)\qquad B_{4,-2}^3 = \begin{bmatrix} u_{-1}^3 & u_{-1}^2 u_{-2} & u_{-1} u_{-2}^2 & u_{-2}^3 \\ 0^3 & 0^2 u_{-1} & 0\cdot u_{-1}^2 & u_{-1}^3 \\ 1^3 & 1^2\,0 & 1\cdot 0^2 & 0^3 \\ 1^3 & 1^2\,1 & 1\cdot 1^2 & 1^3 \end{bmatrix}$$

We transform (2.15) to the desired matrix of (2.12) by

$$B^3_{4,-2}\, Q^3_3 = B^3_{4,1} \tag{2.16}$$

Passing to determinants, and combining (2.16) with (2.14) we have

$$|B^3_{4,1}| = \frac{(-1)^2}{(u_{-2}\, u_{-1})(u_1\, u_2)} \; \frac{|Q_3|^3}{|Q_4|^3} \; |B_{5,1}| \; .$$

Using $u_{-n} = -(-1)^n u_n$ for the negative half of the Fibonacci sequence, and evaluating the known determinants,

$$|B^3_{4,1}| = (-1)^{12}\, S_5\, S_3\, F_4(S) \; \frac{W_4(u)}{F_2(u)\, F_2(u)}$$

The general case, using this technique, may be formulated as

$$|B^r_{n,1}| = (-1)^{nr}\, S_{n+1}\, S_r\, F_n(S) \; \frac{W_n(u)}{F_{r-1}(u)\, F_{n+1-r(u)}} \; . \tag{2.17}$$

Two simplifications to (2.17) are in order:

$$(-1)^{nr}\, S_{n+1}\, S_r = S_{n+1-r}$$

and

$$\frac{W_n(u)}{F_{r-1}(u)\, F_{n+1-r}(u)} = W_{n-1}(u) \; \frac{F_n(u)}{F_{r-1}(u)\, F_{n+1-r}(u)}$$

It seems appropriate, since $F_n(u) = u_n\, u_{n-1}\, u_{n-2} \cdots u_2 u_1$ is a factorial type product for the sequence $\{u_n\}$, that we define the specialized "binomial coefficient"

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{F_n(u)}{F_r(u)F_{n-4}(u)} \; ; \quad F_0(u) = 1, \begin{bmatrix} n \\ r \end{bmatrix} = 1 \; .$$

We then have

$$|B^r_{n,1}| = F_n(S)\; W_{n-1}(u)\; S_{n+1-r} \begin{bmatrix} n \\ r-1 \end{bmatrix} \; .$$

The remaining determinant of (2.10) may now be expanded, and the general case has the form

$$\sum_{r=2}^{n+1} (-1)^r a_r \frac{F_{n+1}(u)}{u_r} \frac{F_n(u)}{u_{r-1}} |B^r_{n-1,1}|$$

or

$$F_{n-1}(S)\; W_n(u) \sum_{r=2}^{n+1} (-1)^r S_{n+1-r} \begin{bmatrix} n+1 \\ r \end{bmatrix} a_r \; .$$

The first two determinants (2.11) round out the summation nicely for $k=1$ and 2, so that we can state

$$D_n\{a\} = F_{n-1}(S)\; W_n(u) \sum_{r=0}^{n+1} (-1)^r S_{n+1-r} \begin{bmatrix} n+1 \\ r \end{bmatrix} a_r$$

or, summing backwards,

$$(2.18) \quad D_n\{a\} = (-1)^{n-1} F_{n-1}(S)\; W_n(u) \sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} a_{n+1-r} \; .$$

At this point we consider the summation

$$(2.19) \quad \phi_n\{a\} = \sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} a_{n+1-r}$$

and the associated polynomial

$$(2.20) \quad \phi_n(x) = \sum_{r=0}^{n+1} (-1)^r S_r \begin{bmatrix} n+1 \\ r \end{bmatrix} x^{n+1-r}$$

For the first few values of n we have

$$
\begin{aligned}
&\phi_0(x) = x-1, \\
&\phi_1(x) = x^2-x-1, \\
&\phi_2(x) = x^3-2x^2-2x+1 = (x+1)(x^2-3x+1), \\
(2.21)\quad &\phi_3(x) = x^4-3x^3-6x^2+3x+1 = (x^2+x-1)(x^2-4x-1), \\
&\phi_4(x) = x^5-5x^4-15x^3+15x^2+5x-1 = (x-1)(x^2+3x+1)(x^2-7x+1) \;.
\end{aligned}
$$

The factorizations suggest the relation

$$(2.22) \qquad \phi_n(x) = (-1)^{n-1}\,(x^2 - v_n x + (-1)^n)\;\phi_{n-2}(-x)$$

where v_n is a Lucas number, and $v_n = u_{n+1} + u_{n-1}$. (2.22) may be proved by induction, and the complete factorization of ϕ_n comes from the identity

$$v_n = a^n + b^n\,, \quad \text{where } a^2 - a - 1 = b^2 - b - 1 = 0 \;.$$

Thus (2.22) becomes

$$\phi_n(x) = (-1)^{n-1}\,(x-a^n)(x-b^n)\;\phi_{n-2}(-x) = (ab)^{n-1}$$

$$\cdot\,(x-a^n)(x-b^n)\;\phi_{n-2}\,(x/ab)\;,$$

and we can construct

$$\phi_n(x) = \prod_{r=0}^{n}\,(x-a^r b^{n-r}) \;.$$

The evaluation of $D_n\{a\}$ for $a_0 = a_1 = \ldots = -1$ becomes, from (2.18) and (2.19), (2.24) $$D_n\{a\} = (-1)^n F_{n-1}(S)\, W_n(u)\, \phi_n(1) .$$

The evaluation of $\phi_n(1)$ requires the investigation of four separate cases. Using (2.23) with $b = -1/a$

(2.25) $$\phi_n(1) = \prod_{r=0}^{n} (1-(-1)^{n-r} a^{2r-n}) = 0 \quad \text{if and only if} \quad n = 4k .$$

When $n = 4k + 2$ the quadratic factorization (2.22) becomes

$$\phi_n(x) = (1+x) \prod_{r=1}^{n/2} (x^2 + (-1)^r v_{2r} + 1)$$

and

$$\phi_n(1) = 2 \prod_{r=1}^{n/2} (v_{2r} + 2(-1)^r) .$$

Using the well known relation $v_r^2 = v_{2r} + 2(-1)^r$ we have

(2.26) $$\phi_n(1) = 2 \prod_{r=1}^{n/2} v_r^2 \quad \text{when} \quad n = 4k + 2 .$$

For $n = 4k \pm 1$ we have, from (2.22)

$$\phi_n(x) = \prod_{r=0}^{\frac{n-1}{2}} (x^2 - (-1)^{r+1} v_{2r+1}\, x - 1) , \quad \text{when} \quad n = 4k - 1$$

$$\phi_n(x) = \prod_{r=0}^{\frac{n-1}{2}} (x^2 + (-1)^{r+1} v_{2r+1}\, x - 1) , \quad \text{when} \quad n = 4k + 1$$

so that

(2.27) $$\phi_n(1) = \prod_{r=0}^{\frac{n-1}{2}} (-1)^r v_{2r-1} = S_{2k-1} \prod_{r=0}^{2k-1} v_{2r+1} ,$$

when $n = 4k - 1$

$$(2.28)\qquad \phi_n(1) = \prod_{r=0}^{\frac{n-1}{2}} (-1)^{r+1} v_{2r+1} = S_{2k+1} \prod_{r=0}^{2k} v_{2r+1} ,$$

$$\text{when} \quad n = 4k + 1 .$$

Combining (2.25), (2.26), (2.27), (2.28) with (2.23), and using the sign convention

$$S_n = (-1)^{n(n-1)/2} \quad \text{and} \quad F_n(S) = -1 \quad \text{only when} \quad n = 4k + 2 ,$$

we have

$$D_{4k} = 0 ,$$

$$D_{4k+1} = (-1)^k W_{4k+1}(u) \prod_{r=0}^{2k} v_{2r+1} ,$$

$$D_{4k-1} = (-1)^k W_{4k-1}(u) \prod_{r=0}^{2k-1} v_{2r+1} ,$$

$$D_{4k+2} = 2\, W_{4k+2}(u) \prod_{r=0}^{2k-1} v_{r+1}^2 ;$$

$$W_n(u) = u_1^n u_2^{n-1} u_3^{n-2} \ldots u_{n-1}^2 u_n = \prod_{r=1}^{n} u_r^{n+1-r} .$$

REFERENCES

1. Brother U. Alfred, Periodic properties of Fibonacci summations, Fibonacci Quarterly, 1(1963), No. 3, pp. 33-42.

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