SQUARE FIBONACCI NUMBERS, ETC.

JOHN H.E. COHN
Bedford College, University of London, London, N.W.1.

INTRODUCTION

An old conjecture about Fibonacci numbers is that 0, 1 and 144 are the only perfect squares. Recently there appeared a report that computation had revealed that among the first million numbers in the sequence there are no further squares [1]. This is not surprising, as I have managed to prove the truth of the conjecture, and this short note is written by invitation of the editors to report my proof. The original proof will appear shortly in [2] and the reader is referred there for details. However, the proof given there is fairly long, and although the same method gives similar results for the Lucas numbers, I have recently discovered a rather neater method, which starts with the Lucas numbers, and it is of this method that an account appears below. It is hoped that the full proof together with its consequences for Diophantine equations will appear later this year. I might add that the same method seems to work for more general sequences of integers, thus enabling equations like $y^2 = Dx^4 + 1$ to be completely solved at least for certain values of D. Of course the Fibonacci case is simply $D = 5$.

PRELIMINARIES

In the first place, in accordance with the practice of the Fibonacci Quarterly, I here use the symbols F_n and L_n to denote the n-th. Fibonacci and Lucas number respectively; in other papers I use the more widely accepted, if less logical, notation u_n and v_n [3]. Throughout the following n, m, k will denote integers, not necessarily positive, and r will denote a non-negative integer. Also, wherever it occurs, <u>k will denote an even integer, not divisible by 3.</u> We shall then require the following formulae, all of which are elementary

(1) $$2F_{m+n} = F_m L_n + F_n L_m$$

(2) $$2L_{m+n} = 5F_mF_n + L_mL_n$$

(3) $$L_{2m} = L_m^2 + (-1)^{m-1}\,2$$

(4) $$(F_{3m}, L_{3m}) = 2$$

(5) $$(F_n, L_n) = 1 \quad \text{if} \quad 3 \nmid n$$

(6) $$2 \mid L_m \quad \text{if and only if} \quad 3 \mid m$$

(7) $$3 \mid L_m \quad \text{if and only if } m \equiv 2 \pmod 4$$

(8) $$F_{-n} = (-1)^{n-1}F_n$$

(9) $$L_{-n} = (-1)^nL_n$$

(10) $$L_k \equiv 3 \pmod 4 \quad \text{if} \quad 2 \mid k,\ 3 \nmid k$$

(11) $$L_{m+2k} \equiv -L_m \pmod{L_k}$$

(12) $$F_{m+2k} \equiv -F_m \pmod{L_k}$$

(13) $$L_{m+12} \equiv L_m \pmod 8$$

THE MAIN THEOREMS

Theorem 1.

If $L_n = x^2$, then $n = 1$ or 3.

Proof.

If n is even, (3) gives

$$L_n = y^2 \pm 2 \neq x^2 \ .$$

If $n \equiv 1 \pmod 4$, then $L_1 = 1$, whereas if $n \neq 1$ we can write $n = 1 + 2\cdot 3^r \cdot k$ where k has the required properties, and then obtain by (11)

$$L_n \equiv -L_1 = -1 \pmod{L_k}$$

and so $L_n \neq x^2$ since -1 is a non-residue of L_k by (10). Finally, if $n \equiv 3 \pmod 4$ then $n = 3$ gives $L_3 = 2^2$, whereas if $n \not\equiv 3$, we write as before $n = 3 + 2 \cdot 3^r \cdot k$ and obtain

$$L_n \equiv -L_3 = -4 \pmod{L_k}$$

and again $L_n \neq x^2$.
This concludes the proof of Theorem 1.

<u>Theorem 2.</u>

If $L_n = 2x^2$, then $n = 0$ or ± 6.

Proof.

If n is odd and L_n is even, then by (6) $n \equiv \pm 3 \pmod{12}$ and so, using (13) and (9),

$$L_n \equiv 4 \pmod 8$$

and so $L_n \neq 2x^2$.

Secondly, if $n \equiv 0 \pmod 4$, then $n = 0$ gives $L_n = 2$, whereas if $n \neq 0$, $n = 2 \cdot 3^r \cdot k$ and so

$$2L_n \equiv -2L_0 = -4 \pmod{L_k}$$

whence $$2L_n \neq y^2, \text{ i.e. } L_n \neq 2x^2$$

Thirdly, if $n \equiv 6 \pmod 8$ then $n = 6$ gives $L_6 = 2 \cdot 3^2$ whereas if $n \neq 6$, $n = 6 + 2 \cdot 3^r \cdot k$ where now $4 | k$, $3 \nmid k$ and so

$$2L_n \equiv -2L_6 = -36 \pmod{L_k}$$

and again, -36 is a non-residue of L_k using (7) and (10). Thus as before $L_n \neq 2x^2$.

Finally, if $n \equiv 2 \pmod 8$, then by (9) $L_{-n} = L_n$ where now $-n \equiv 6 \pmod 8$ and so the only admissible value is $-n = 6$, i.e. $n = -6$.

This concludes the proof of Theorem 2.

Theorem 3.

If $F_n = x^2$, then $n = 0, \pm 1, 2$ or 12.

Proof.

If $n \equiv 1 \pmod 4$, then $n = 1$ gives $F_1 = 1$, whereas if $n \neq 1$, $n = 1 + 2 \cdot 3^r \cdot k$ and so

$$F_n \equiv - F_1 = - 1 \pmod{L_k}$$

whence $F_n \neq x^2$. If $n \equiv 3 \pmod 4$, then by (8) $F_{-n} = F_n$ and $-n \equiv 1 \pmod 4$ and as before we get only $n = -1$. If n is even, then by (1) $F_n = F_{\frac{1}{2}n} L_{\frac{1}{2}n}$ and so, using (4) and (5) we obtain, if $F_n = x^2$

either $3 | n$, $F_{\frac{1}{2}n} = 2y^2$, $L_{\frac{1}{2}n} = 2z^2$. By Theorem 2, the latter is possible only for $\frac{1}{2}n = 0$, 6 or -6. The first two values also satisfy the former, while the last must be rejected since it does not.

or $3 \nmid n$, $F_{\frac{1}{2}n} = y^2$, $L_{\frac{1}{2}n} = z^2$. By Theorem 1, the latter is possible only for $\frac{1}{2}n = 1$ or 3, and again the second value must be rejected.

This concludes the proof of Theorem 3.

Theorem 4.

If $F_n = 2x^2$, then $n = 0, \pm 3$ or 6.

Proof.

If $n \equiv 3 \pmod 4$, then $n = 3$ gives $F_3 = 2$, whereas if $n \neq 3$, $n = 3 + 2 \cdot 3^r \cdot k$ and so

$$2F_n \equiv - 2F_3 = -4 \pmod{L_k}$$

and so $F_n \neq 2x^2$. If $n \equiv 1 \pmod 4$ then as before $F_{-n} = F_n$ and we get only $n = -3$. If n is even, then since $F_n = F_{\frac{1}{2}n} L_{\frac{1}{2}n}$ we must have if $F_{\frac{1}{2}n} = 2x^2$

either $F_{\frac{1}{2}n} = y^2$, $L_{\frac{1}{2}n} = 2z^2$; then by Theorems 2 and 3 we see that the only value which satisfies both of these is $\frac{1}{2}n = 0$

or $F_{\frac{1}{2}n} = 2y^2$, $L_{\frac{1}{2}n} = z^2$; then by Theorem 1, the second of these is satisfied only for $\frac{1}{2}n = 1$ or 3. But the former of these does not satisfy the first equation.

This concludes the proof of the theorem.

REFERENCES

1. M. Wunderlich, On the non-existence of Fibonacci Squares, Maths. of Computation, 17 (1963) p. 455.
2. J. H. E. Cohn, On Square Fibonacci Numbers, Proc. Lond. Maths. Soc. 39 (1964) to appear.
3. G. H. Hardy and E. M. Wright, Introduction to Theory of Numbers, 3rd. Edition, O. U. P. 1954, p. 148 et seq.

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EDITORIAL NOTE

Brother U. Alfred cheerfully acknowledges the priority of the essential method, used in "Lucas Squares" in the last issue of the Fibonacci Quarterly Journal, rest solely with J. H. E. Cohn. This was written at the request of the Editor and the unintentional omission of due credit rest solely with the Editor.