ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A.P. HILLMAN
University of Santa Clara, Santa Clara, California

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer should submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated.

Solutions to problems listed below should be submitted within two months of publication.

B-38 *Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California*

Characterize simply all the sequences c_n satisfying

$$c_{n+2} = 2c_{n+1} - c_n \ .$$

B-39 *Proposed by John Allen Fuchs, University of Santa Clara, Santa Clara, California*

Let $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Prove that

$$F_{n+2} < 2^n \text{ for } n \geq 3 \ .$$

B-40 *Proposed by Charles R. Wall, Texas Christian University, Fort Worth, Texas*

If H_n is the n-th term of the generalized Fibonacci sequence, i.e.,

$$H_1 = p, \; H_2 = p + q, \; H_{n+2} = H_{n+1} + H_n \text{ for } n \geq 1 ,$$

show that

$$\sum_{k=1}^{n} kH_k = (n + 1)H_{n+2} - H_{n+4} + 2p + q .$$

B-41 *Proposed by David L. Silverman, Beverly Hills, California*

Do there exist four distinct positive Fibonacci numbers in arithmetic progression?

B-42 *Proposed by S.L. Basin, Sylvania Electronic Systems, Mountain View, California*

Express the $(n + 1)$-st Fibonacci number F_{n+1} as a function of F_n. Also solve the same problem for Lucas numbers.

B-43 *Proposed by Charles R. Wall, Texas Christian University, Fort Worth, Texas*

(a) Let $x_0 \geq 0$ and define a sequence x_k by $x_{k+1} = f(x_k)$ for $k \geq 0$, where $f(x) = \sqrt{1 + x}$. Find the limit of x_k as $k \to \infty$.

(b) Solve the same problem for $f(x) = \sqrt[3]{1 + 2x}$.

(c) Solve the same problem for $f(x) = \sqrt[4]{2 + 3x}$.

(d) Generalize.

SOLUTIONS

FIBONACCI AND PASCAL AGAIN

B-16 *Proposed by Marjorie Bicknell, San Jose State College, San Jose, California, and Terry Brennan, Lockheed Missiles and Space Co., Sunnyvale, California*

Show that if

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

then

$$R^n = \begin{pmatrix} F_{n-1}^2 & 2F_{n-1}F_n & F_n^2 \\ F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & F_nF_{n+1} \\ F_n^2 & 2F_nF_{n+1} & F_{n+1}^2 \end{pmatrix} .$$

(There are some misprints in the original statement.)

Solution by L. Carlitz, Duke University, Durham, N.C.

Put

$$R_k = \left[\binom{r}{k-s} \right] \qquad (r,\ s = 0,\ 1,\ \ldots,\ k) ,$$

a matrix of order $k + 1$; for example

$$R_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

It is easily verified that

$$(1) \qquad R_1^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \qquad (n = 1,\ 2,\ \ldots) .$$

Indeed this is obviously true for $n = 1$. Assuming that the formula holds for n, we have

$$R_1^{n+1} = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{bmatrix} .$$

In the next place we notice that the transformation

$$T_1: \begin{cases} x' = y \\ y' = x + y \end{cases}$$

induces the transformations

$$T_2: \begin{cases} x'^2 = y^2 \\ x'y' = xy + y^2 \\ y'^2 = x^2 + 2xy + y^2 \end{cases},$$

$$T_3: \begin{cases} x'^3 = y^3 \\ x'^2y' = xy^2 + y^3 \\ x'y'^2 = x^2y + 2xy^2 + y^3 \\ y'^3 = x^3 + 3x^2y + 3xy^2 + y^3 \end{cases}$$

and so on. Also it is evident from (1) that T_1^n is given by

$$T_1^n: \begin{cases} x^{(n)} = F_{n-1}x + F_n y \\ y^{(n)} = F_n x + F_{n+1} y \end{cases}.$$

We therefore get

$$T_1^n: \begin{cases} (x^{(n)})^2 = F_{n-1}^2x^2 + 2F_{n-1}F_n xy + F_n^2y^2 \\ x^{(n)}y^{(n)} = F_{n-1}F_n x^2 + (F_n^2 + F_{n-1}F_{n+1})xy + F_nF_{n+1}y^2 \\ (y^{(n)})^2 = F_n^2x^2 + 2F_nF_{n+1}xy + F_{n+1}^2y^2 \end{cases}.$$

Also solved by the proposers.

LAMBDA FUNCTION OF A MATRIX

B-24 *Proposed by Brother U. Alfred, St. Mary's College, California*

It is evident that the determinant

$$\begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_{n+1} & F_{n+2} & F_{n+3} \\ F_{n+2} & F_{n+3} & F_{n+4} \end{vmatrix}$$

has a value of zero. Prove that if the same quantity k is added to each element of the above determinant, the value becomes $(-1)^{n-1}k$.

Solution by Raymond Whitney, Pennsylvania State University, Hazelton Campus

Using the basic Fibonacci recursion formula $F_{n+2}=F_{n+1}+F_n$ and elementary row and column transformations we may reduce the determinant to:

$$= k \begin{vmatrix} F_n & F_{n+1} & -1 \\ F_{n+1} & F_{n+2} & -1 \\ 0 & 0 & 1 \end{vmatrix} = k(F_nF_{n+2} - F^2_{n+1}) ,$$

which is $(-1)^{n-1}k$ by a basic identity.

Also solved by Marjorie Bicknell, San Jose State College, San Jose, California who pointed out the relation to "Fibonacci Matrices and Lambda Functions," by M. Bicknell and V. E. Hoggatt, Jr., this Quarterly, Vol. 1, No. 2; R. M. Grassl, University of Santa Clara, California; *F.D. Parker, State University of New York, Buffalo, N.W., R.N. Vawter, St. Mary's College, California; H.L. Walton, Yorktown H.S., Arlington, Virginia; and the proposer.*

EXPONENTIALS OF FIBONACCI NUMBERS

B-25 *Proposed by Brother U. Alfred, St. Mary's College, California*

Find an expression for the general term(s) of the sequence $T_0 = 1$, $T_1 = a$, $T_2 = a$, ... where

$$T_{2n} = \frac{T_{2n-1}}{T_{2n-2}} \quad \text{and} \quad T_{2n+1} = T_{2n}T_{2n-1} .$$

Solution by Vassili Daiev, Sea Cliff, L.I., N.Y.

The first few terms are

$$a^{F_0},\ a^{F_2},\ a^{F_1},\ a^{F_3},\ a^{F_2},\ a^{F_4},\ \ldots\ .$$

It is easy to see that $T_n = a^k$ where $k = F_{(n/2)}$ if n is even and $k = F_{(n+3)/2}$ if n is odd.

Also solved by J.A.H. Hunter, Toronto, Ontario, Canada who suggested the consideration of $\log T_n$; Ralph Vawter, St. Mary's College, California, and the proposer.

Editorial Comment: The problem can be solved by showing that $\log T_{m+4} = \log T_{m+2} + \log T_m$.

MAXIMIZING A DETERMINANT

B-28 *Proposed by Brother U. Alfred, St. Mary's College, California*

Using the nine Fibonacci numbers F_2 to F_{10} (1, 2, 3, 5, 8, 13, 21, 34, 55), determine a third-order determinant having each of these numbers as elements so that the value of the determinant is a maximum.

Solution by Marjorie Bicknell, San Jose State College, California

By considering combinations of Fibonacci numbers which give minimum and maximum values to sums of the form abc + def + ghi, the following determinant seems to have the maximum value obtainable with the nine Fibonacci numbers given:

$$\begin{vmatrix} F_{10} & F_4 & F_7 \\ F_6 & F_9 & F_3 \\ F_2 & F_5 & F_8 \end{vmatrix} = F_{10}F_9F_8 + F_7F_6F_5 + F_4F_3F_2 - (F_{10}F_3F_5 + F_9F_2F_7 + F_8F_4F_6)$$

$$= 39796 - 1496$$

$$= 38300.$$

Also solved by the proposer.

B-29 *Proposed by A.P. Boblett, U.S. Naval Ordnance Laboratory, Corona, California*

Define a general Fibonacci sequence such that

$$F_1 = a; \quad F_2 = b; \quad F_n = F_{n-2} + F_{n-1}, \quad n \geq 3$$
$$F_n = F_{n+2} - F_{n+1}, \quad n \leq 0$$

Also define a characteristic number, C, for this sequence, where $C = (a + b)(a - b) + ab$.

Prove:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n C, \text{ for all } n .$$

Solution by F.D. Parker, State University of New York, Buffalo, N.Y.

From $F(n) = F(n - 1) + F(n - 2)$, $F(1) = a$, $F(2) = b$, we get

$$F(n) = \frac{b - ar}{1 + s^2} s^n + \frac{b - as}{1 + r^2} r^n,$$

where r and s are solutions of the quadratic $x^2 - x - 1 = 0$. Using the fact that $r + s = -rs = 1$, direct calculation yields

$$F(n + 1)F(n - 1) - F^2(n) = [(a - b)(a + b) + ab] (-1)^n .$$

The well known result $F(n + 1)F(n - 1) - F^2(n) = (-1)^n$ is the special case in which $a = b = 1$.

Also solved by Marjorie Bicknell, San Jose State College, San Jose, California; Donna J. Seaman, Sylvania Co.; R.N. Vawter, St. Mary's College, California; and the proposer, J.A.H. Hunter, of Toronto, Ontario.