

## A MOTIVATION FOR CONTINUED FRACTIONS

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This Quarterly is devoted to the study of properties of integers, especially to the study of recurrent sequences of integers. We show below how such sequences and continued fractions arise naturally in the problem of approximating an irrational number to any desired closeness by rational numbers.

We begin with the equation

$$(1) \quad x^2 - x - 1 = 0 .$$

One can easily see that there is a negative root between -1 and 0 and a positive root between 1 and 2, for example by graphing  $y = x^2 - x - 1$ . We call the positive root  $r$ . This number has been known since antiquity as the "golden mean." We now look for a sequence of rational approximations to  $r$ .

A rational number is of the form  $p/q$  with  $p$  and  $q$  integers (and  $q \neq 0$ ). We therefore wish two sequences

$$(2) \quad \begin{array}{l} p_1, p_2, p_3, \dots \\ q_1, q_2, q_3, \dots \end{array}$$

of integers such that the quotients  $p_n/q_n$  are approximations which get arbitrarily close to  $r$ . It would also be helpful if each new approximation were obtainable simply from previous ones.

We go back to equation (1) and rewrite it as

$$(3) \quad x = 1 + \frac{1}{x} .$$

This states that if we replace  $x$  by  $r$  in

$$(4) \quad 1 + \frac{1}{x}$$

the result is  $r$  and suggests that if we replace  $x$  in (4) by an approximation to  $r$  we will get another approximation. We now change (3) into the form

$$(5) \quad x_2 = 1 + \frac{1}{x_1}$$

and consider  $x_1$  to be an approximation to  $r$ . The relative error of  $1/x_1$  is the same as that of  $x_1$  and, if  $x_1$  is positive, the relative error of  $x_2$  (i. e.,  $1 + 1/x_1$ ) is lower than that of  $x_1$ , since adding 1 increases the number but not the error. It can be shown that  $x_2$  in (5) is a better approximation to  $r$  than  $x_1$ , if  $x_1 > 0$ .

We now let our first approximation  $x_1$  be a rational number  $p_1/q_1$  and substitute this in (5) obtaining

$$x_2 = 1 + \frac{1}{(p_1/q_1)} = 1 + \frac{q_1}{p_1} = \frac{p_1 + q_1}{p_1}.$$

We therefore choose  $p_2$  to be  $p_1 + q_1$  and  $q_2$  to be  $p_1$ . Similarly, our third approximation is  $p_3/q_3$  with  $p_3 = p_2 + q_2$  and  $q_3 = p_2$ . In general, the  $(n+1)$ -st approximation  $p_{n+1}/q_{n+1}$  has

$$(6) \quad p_{n+1} = p_n + q_n$$

$$(7) \quad q_{n+1} = p_n.$$

It follows from (7) that  $q_n = p_{n-1}$ ; substituting this in (6) gives

$$(8) \quad p_{n+1} = p_n + p_{n-1}.$$

Since  $r$  is between 1 and 2 we use 1 as the first approximation, i. e., we let  $p_1 = q_1 = 1$ . This means that  $p_2 = 2$  and it now follows from (8) that  $p_n$  is the Fibonacci number  $F_{n+1}$ . Then (7) implies that  $q_n = F_n$  and we see that the sequence of quotients  $F_{n+1}/F_n$  of consecutive Fibonacci numbers furnishes the desired approximations to the root  $r$  of (1). It can be shown that this sequence converges to  $r$  in the calculus sense.

We next consider the problem of approximating  $s = \sqrt{10}$  in this way. The number  $s$  is the positive root of

$$(9) \quad x^2 - 10 = 0 .$$

We write (9) in the forms

$$(10) \quad \begin{aligned} x^2 - 9 &= 1 \\ (x - 3)(x + 3) &= 1 \\ (x - 3) &= 1/(x + 3) \\ x &= 3 + 1/(x + 3) \end{aligned}$$

and change (10) into

$$(11) \quad x_{n+1} = 3 + \frac{1}{3 + x_n} .$$

Again, if  $x_n$  is a positive approximation to  $s$ , it can be seen that  $x_{n+1}$  is an approximation with smaller relative error. There is a sequence of rational approximations  $p_n/q_n$  with

$$p_{n+1} = 3p_n + 10q_n, \quad q_{n+1} = p_n + 3q_n .$$

Letting the first approximation be 3, i. e., letting  $p_1 = 3$  and  $q_1 = 1$ , we obtain the sequence

$$3/1, 19/6, 117/37, \dots$$

which can be shown to converge to  $s$ .

Equation (11) contains the equations

$$x_2 = 3 + \frac{1}{3 + x_1}, \quad x_3 = 3 + \frac{1}{3 + x_2} .$$

Substituting the first of these into the second gives us

$$x_3 = 3 + \frac{1}{6 + \frac{1}{3 + x_1}}$$

If this is substituted into  $x_4 = 3 + 1/(3 + x_3)$  and if we let  $x_1$  be 3, we obtain

$$x_4 = 3 + \frac{1}{6 + \frac{1}{6 + \frac{1}{6}}}$$

In this way we can write continued fraction expressions for any one of the  $x_n$ . Then it is natural to let the infinite continued fraction

$$3 + \frac{1}{6 + \frac{1}{6 + \dots}}$$

represent the limit  $s$  of the sequences  $x_n$  defined by (11) and  $x_1 = 3$ .

The infinite continued fraction for the root  $r$  of  $x^2 - x - 1 = 0$  is

$$1 + \frac{1}{1 + \frac{1}{1 + \dots}},$$

whose elegant simplicity is worthy of the title "golden mean."

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#### FIBONACCI AND LUCAS NUMBER TABLES

Those interested may secure bound mimeographed tables of the first 1505 Fibonacci numbers,  $F_n$ , and the first 1506 Lucas numbers,  $L_n$ , by sending two dollars to Professor Jack K. Ward, Westminster College, Fulton, Missouri.