EXPLORING THE FIBONACCI REPRESENTATION OF INTEGERS

Proposed by Brother U. Alfred on page 72, Dec. 1963, "The Fibonacci Quarterly"

The completion of the Theorem stated in the article is:
The Maximum number of different Fibonacci numbers required to represent an integer N for which $[N]^* = F_n$ is given by $\left[\frac{n}{2}\right]$.

This is a corollary of the following theorem.
For $F_n < N \le F_{n+1}$ the number N can be represented as a sum of Fibonacci numbers, the largest which is F_n and the smallest greater than or equal to F_2. Moreover, the sum never contains two consecutive Fibonacci numbers. We therefore have at most the alternating terms of indices from 2 to n which gives us $\left[\frac{n-2}{2} + 1\right] = \left[\frac{n}{2}\right]$, as claimed.

The proof of this theorem depends upon a Lemma which is a well known Fibonacci Identity that $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$ and that $F_3 + F_5 + F_7 + \ldots + F_{2n-1} = F_{2n} - 1$. The proof of the first part of this is given by induction and the second part is similarly proved.
Proof.
For n = 1, we have $F_2 = F_3 - 1$

n = 2, we have $F_2 + F_4 = F_5 - 1$ which clearly shows the Lemma holds for n = 1, 2.

Now assume that it holds for all $n \le K$, where K is a fixed but unspecified positive integer greater than or equal to 3.
i.e. $F_2 + F_4 + \ldots + F_{2K} = F_{2K+1} - 1$, therefore by addition to both sides we have that

$$F_2 + F_4 + \ldots + F_{2K} + F_{2K+2} = F_{2K+1} + F_{2K+2} - 1 = F_{2K+3} - 1$$

which implies the Lemma holds for all positive n.

Using this Lemma which we shall call Lemma 1, part A for the first part which was just proved, and part B for the second part with the odd indices; we can now prove the general theorem that for $F_n < N \le F_{n+1}$, we can represent N as a sum of at least alternating Fibonacci numbers where the largest is F_n for $N < F_{n+1}$ and which trivially is just F_{n+1} itself when $N = F_{n+1}$.
Proof. For N = 1, we have $1 = F_2$, and for N = 2, we have $2 = F_3$. Now assume the theorem true for all $N \le k$, where k is a fixed but unspecified positive integer and n is such that $F_n < k \le F_{n+1}$, $n \ge 3$. Now if

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