# AMATEUR INTERESTS IN THE FIBONACCI SERIES – PRIME NUMBERS

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My interest in the Fibonacci series was born in 1959 when it was noticed that the preferred ratios developed in the research of my colleague, H. Ellner, and later included in Department of Defense Handbook H109 [1], were 1, 2, 3, 5 and 8. From recollection of a brief mention in college algebra, this was recognized as the first few terms of the Fibonacci. To test the supposition that the preferred ratios would all be from this series, the next one was calculated and, sure enough, it was 13. Then it was noted that the sample sizes, Acceptable Quality Levels (AQL's) and lot size ranges of all sampling standards since Dodge and Romig [2] were series approximately of the type:

(1) $$u_{n+2} = u_{n+1} + u_n$$

In fact the latest version of Military Standard Mil Std 105 [3] shows sample sizes which are almost exactly the Fibonacci series itself. These occurrences were too remarkable to be ascribed to mere coincidence and my interest led me to examine the series empirically. According to Dickson [4], the literature on this subject is rich, extending as it does from the year 1202 to the present. However, it is almost completely unavailable to me and, I suspect, to most others.

On developing the series $u_n$ from $n = 0$ to $n = 25$ or so, inspection soon revealed that two thirds of the series comprised odd numbers and exactly every third $u_n$ was even. It did not take much to ascertain why this is so. In this way I found that $n$, the ordinal of $u_n$ in the series was, in a manner of speaking, the determinant of the properties of $u_n$. Thus, if $z$ is a factor of $u_n$ it will infallibly be a factor of $u_{2n}$, $u_{3n}$, etc. Therefore, in general, if $n$ is composite, so is $u_n$ (except for the case $n = 4$, $u_n = 3$), but if $n$ is prime, $u_n$ may be prime. My first guess that, since the density of odd numbers in the Fibonacci is twice that of the even numbers, the density of primes would be greater than in the cardinal number domain was proven wrong when the primality of $u_n$ was found dependent on $n$ being prime. The next supposition of equal density was shown to be wrong when $u_{31} = 1346269$ was found to be a composite of 557 and 2417. When $u_{37}$ and $u_{41}$ were also determined to be composite it became obvious that the density of primes in $u_n$ was less than that of the cardinal domain.

Several other interesting details were elucidated after extending and examining the series, first down to $n = 50$ then to $n = 100$ and finally to $n = 130$. No $u_n$ is divisible by $n$ except when $n = 5$ or powers of 5. For example $u_5 = 5$ and $u_{25} = 75025$. Except for $u_6$, every $u_n$ seems to have at least one prime factor which has not been a factor of any previous $u_n$; some have two or three such new prime factors. Surely, any theory of prime numbers might profit from Fibonacci considerations.

However, the first gain from the extension of study of the series to $n = 100$ was a remarkable regularity found from the fact that if $P_n$ is a prime factor of $u_n$ it will also factor, more generally, $u_{jn}$ where $j$ goes from 1 to 00. Consider the multiple $j$ and let this be expressed as a sum of multiples of powers of $P_n$, reduced to a minimum of terms, and provided that no multiples of the powers of $P_n \geq P_n$. Thus:

$$(2) \qquad j = aP_n^0 + bP_n^1 + cP_n^2 + \ldots qP_n^r$$

where a, b, c ... q may be zero but must always be less than $P_n$. Then $u_{jn}$ will be divisible by $P_n^{x+1}$ where $P_n^x$ is the lowest power term of $P_n$ in the sum of multiples of powers of $P_n = j$.

Example 1.

The first prime to divide $u_n$ is 2 ($P_n = 2$) and it divides the third number ($n = 3$) in the series: $u_3 = 2$. From the above lemma we have:

| Ordinal jn | j | Sum of $P_n$* terms = j | $P_n^x$ x | $u_{jn}$ is divisible by $P_n^{x+1}$ | $u_{jn}$ |
|---|---|---|---|---|---|
| 3 | 1 | $P_n^0$ | 0 | $P_n^{0+1} = P_n^1 = 2$ | 2 |
| 6 | 2 | $P_n^1$ | 1 | $P_n^{1+1} = P_n^2 = 4$ | 8 |
| 9 | 3 | $P_n^0 + P_n^1$ | 0 | $P_n^{0+1} = P_n^1 = 2$ | 34 |
| 24 | 8 | $P_n^3$ | 3 | $P_n^{3+1} = P_n^4 = 16$ | 46368 |
| 30 | 10 | $P_n^1 + P_n^3$ | 1 | $P_n^{1+1} = P_n^2 = 4$ | 832040 |
| 33 | 11 | $P_n^0 + P_n^1 + P_n^3$ | 0 | $P_n^{0+1} = P_n^1 = 2$ | 3524578 |

*Since $P_n = 2$, no multiples other than 0 or 1 appear in the sum of powers of $P_n = j$. Actually the sum of multiples of power terms for $j = 11$ should read:

$$11 = 1P_n^0 + 1P_n^1 + 0P_n^2 + 1P_n^3 = P_n^0 + P_n^1 + P_n^3 = 2^0 + 2^1 + 2^3 = 1 + 2 + 8.$$

Example 2.

Another prime dividing $u_n$ is 5 ($P_n = 5$) and, as already mentioned, it divides the fifth number in the series: $u_5 = 5$. Again we make a table:

| Ordinal jn | j | Sum of $P_n$ terms = j | $P_n^x$ x | $u_{jn}$ is divisible by $P_n^{x+1}$ | $u_{jn}$ |
|---|---|---|---|---|---|
| 5 | 1 | $P_n^0$ | 0 | $P_n^{0+1} = P_n^1 = 5$ | 5 |
| 10 | 2 | $2P_n^{0*}$ | 0 | $P_n^{0+1} = P_n^1 = 5$ | 55 |
| 20 | 4 | $4P_n^0$ | 0 | $P_n^{0+1} = P_n^1 = 5$ | 6765 |
| 25 | 5 | $P_n^1$ | 1 | $P_n^{1+1} = P_n^2 = 25$ | 75025 |
| 30 | 6 | $P_n^0 + P_n^1$ | 0 | $P_n^{0+1} = P_n^1 = 5$ | 832040 |
| 35 | 7 | $2P_n^0 + P_n^1$ | 0 | $P_n^{0+1} = P_n^1 = 5$ | 9227465 |
| 50 | 10 | $2P_n^1$ | 1 | $P_n^{1+1} = P_n^2 = 25$ | 12586269025 |
| 125 | 25 | $P_n^2$ | 2 | $P_n^{2+1} = P_n^3 = 125$ | 59425114757512643212875125 |

*The multiple 2 of $2P_n^0$ plays no part, only the power of $P_n$ (zero in this case) is used.

At a later time, in a private communication, Dr. S. M. Ulam recommended Dickson [4] as a reference to the literature. In this I discovered that these findings were known to Lucas [5]. In particular, according to Dickson, the above was stated by Lucas as Theorem V of eight in the following form:

"If n is the rank of the first term $u_n$ containing the prime factor p to the power $\lambda$, then $u_{pn}$ is the first term divisible by $p^{\lambda+1}$ and not by $p^{\lambda+2}$; this is called the law of repetition of primes in the recurring series of $u_n$."

On reading this it is clear that precedence in this finding lay with Lucas who had, moreover, stated it more clearly and economically. Far from being discouraged, however, I continued my search, listing all prime numbers up to 10009 and laboriously testing the primality of most $u_n$'s up to $n = 130$. Of course, primes up to 10009 are sufficient only to test $u_n$ up to $n = 40$ directly but the fact that if z divides $u_n$ it will divide $u_{jn}$ helped greatly. Nevertheless it speedily became apparent that repeated division of $u_n$ greater than $u_{45}$ on a desk calculator was not only laborious but increasingly prone to error as the number of digits in $u_n$ rose above 10. If only there were some way to eliminate some of the trial divisions!

A study of the primes, $P_n$, which divide $u_n$ revealed that they were all of the form

$$P_n = an + 1 \tag{3}$$

Since $P_n$ is prime it is obvious that an had to be even so that $an \pm 1$ could be odd. Therefore either a or n or both had to be even. Closer study of the primes indicated that, when a and n were both even, it was always necessary to add one to an to get $P_n$, i.e. with a and n even, $P_n = an + 1$, never $an - 1$. I cannot explain this but, empirically, it turns out this way. Now it was possible to cut down on the number of divisions required to determine the $P_n$ which would divide $u_n$. Thus:

a. Calculate $2n + 1$ (If $n$ is even, determine only $2n + 1$).

b. Determine whether $2n + 1$ and/or $2n - 1$ are prime.

c. Divide $u_n$ by any prime number determined in a and b.

d. If $u_n$ is not divided in c, calculate $3n + 1$.

e. Repeat setps c. and d.

f. If $u_n$ is not divided in step e, calculate $4n + 1$ If n is even, determine $4n + 1$ only).

g. Continue until the $P_n$ which divides $u_n$ is found.

The relationship found above may be expressed as follows:

If P is any prime there exists an n such that $P_n = an + 1$ or $an - 1$ will divide $u_n$ without remainder (a being some whole number $> 0$.). The only exception is $P_n = 5$ which divides $u_5 = 5$.

It is possible that the above relationship would repay investigation in prime number theory. In the past, a number of formulas have been proposed for the purpose of generating prime numbers. In every case the formulas have been found faulty in one or more of the following respects:

a. The density of primes generated has been much lower than the true density of primes.

b. They have generated composite numbers.

c. They have rarely been capable of generating paired primes (two consecutive primes which differ by 2, e.g. 11 and 13).

The formula given in (3) suffers only in generating very many composites. However, the procedure clearly furnishes a criterion whereby (empirically) it has been found that, if n is prime, $u_n$ will be divided by $P_n$ only when $P_n$, determined as in (3), is prime. If this can be proven, new light may be shed by the proof on this age-old problem.

## REFERENCES


1. Department of Defense Handbook H109 "Statistical Procedures for Determining Validity of Suppliers' Attributes Inspection," 6 May 1960.
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3. Military Standard Mil Std 105D, "Sampling Procedures and Tables for Inspection by Attributes," in process of publication by Department of Defense. Revision Mil Std 105C is dated 18 July 1961.
4. Dickson, "History of the Theory of Numbers," Chapter XVII, Recurring Series, Lucas' $u_n$, $v_n$.
5. Lucas, "Sur la théorie des nombres premiers," Atti R. Accad. Sc. Torino (Math.), 11, 1875-6, 928-937, cited in [4].


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