ON SUMMATION FORMULAS FOR FIBONACCI AND LUCAS NUMBERS

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Recently, Siler [1] gave a closed form for $\sum_{k=1}^{n} F_{ak-b}$, where $a > b$ are positive integers and F_k are Fibonacci numbers with $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$, $n = 0, 1, \ldots$. In this note, we will establish a more general summation formula which yields the result of [1] as a special case. General summation formulas for Fibonacci and Lucas numbers will be obtained as special cases of our general result.

<u>Theorem</u>. Let p, q, u_o, and u_1 be arbitrary real numbers, and let

(1) $$u_{n+2} = q\,u_{n+1} - pu_n \qquad (n = 0, 1, \ldots),$$

(2) $$S_n = r_1^n + r_2^n \qquad (n = 0, 1, \ldots),$$

where $r_1 \neq r_2$ are roots of $x^2 - qx + p = 0$ (i.e., $q^2 - 4p \neq 0$). We define

(3) $$u_{-n} = (u_0 S_n - u_n)/p^n \qquad (n = 1, 2, \ldots),$$

(4) $$S_{-n} = S_n/p^n \qquad (n = 1, 2, \ldots).$$

Let $a = 0, 1, \ldots$; $d = 0, \pm 1, \pm 2, \ldots$, and let x be a real number. Then

(5) $$(1 - S_a x + p^a x^2) \sum_{k=0}^{n} u_{ak+d} x^k = p^a x^{n+2} u_{an+d} - x^{n+1} u_{an+a+d} + x u_{a+d} + (1 - xS_a) u_d \ .$$

Moreover, in the region of convergence, we have

(6) $$(1 - S_a x + p^a x^2) \sum_{k=0}^{\infty} u_{ak+d} x^k = u_d + (u_{a+d} - u_d S_a)\, x \ .$$

<u>Proof</u>. If C_i, $i = 1, 2$, are arbitrary constants, then $u_n = C_1 r_1^n + C_2 r_2^n$, $n = 0, 1, \ldots$, is the general solution of (1). Then

$$v_k = u_{ak+d} = (C_1 r_1^d)(r_1^a)^k + (C_2 r_2^d)(r_2^a)^k, \quad k = 0, 1, \ldots,$$

satisfies the linear difference equation

$$(7) \qquad v_{k+2} = S_a v_{k+1} - p^a v_k \qquad (k = 0, 1, \ldots) ,$$

since $(x^2 - S_a x + p^a) \equiv (x - r_1^a)(x - r_2^a)$. Let

$$g(x) = \sum_{k=0}^{n} v_k x^k .$$

Multiplying both sides of (7) by x^{k+2} and then summing both sides with respect to k, we obtain

$$(8) \qquad \sum_{k=0}^{n} v_{k+2} x^{k+2} = xS_a \sum_{k=0}^{n} v_{k+1} x^{k+1} - p^a x^2 \sum_{k=0}^{n} v_k x^k .$$

We note that

$$(9) \qquad \sum_{k=0}^{n} v_{k+2} x^{k+2} = g(x) + v_{n+2} x^{n+2} + v_{n+1} x^{n+1} - v_1 x - v_0 ,$$

$$(10) \qquad \sum_{k=0}^{n} v_{k+1} x^{k+1} = g(x) + v_{n+1} x^{n+1} - v_0 .$$

If we substitute (9) and (10) into (8), use (7) to eliminate v_{n+2}, and solve for g(x), we obtain our principal result, (5).

The generating function for v_k is readily obtained from (5). Let $R > 0$ and suppose that $\sum_{k=0}^{\infty} v_k x^k$ converges for $|x| < R$. Then, for $|x| < R$, $v_n x^n \to 0$ as $n \to \infty$. Thus, for $|x| < R$, (5) yields (6) as $n \to \infty$.

Remarks. Let q = 1 and p = -1 in (1). Then, for $u_0 = 0$ and $u_1 = 1$, we have $u_n \equiv F_n$, $F_{-n} = (-1)^{n+1} F_n$, and $S_n \equiv L_n$, the well-known Lucas sequence, where $L_0 = 2$ and $L_1 = 1$. Thus, (5) and (6), for $u_n \equiv F_n$, become, respectively,

$$(11) \qquad \begin{aligned} (1 - L_a x + (-1)^a x^2) \sum_{k=0}^{n} F_{ak+d} x^k &= (-1)^a x^{n+2} F_{an+d} \\ &\quad - x^{n+1} F_{an+a+d} + xF_{a+d} + (1 - xL_a) F_d , \end{aligned}$$

$$(12) \qquad (1 - L_a x + (-1)^a x^2) \sum_{k=0}^{\infty} F_{ak+d} x^k = F_d + (F_{a+d} - F_d L_a) x .$$

The main result of [1] is obtained from (11) for $x = 1$ and $d = -b$.

For $x = -1$, (11) yields the interesting result

$$(1+(-1)^a+L_a)\sum_{k=0}^{n}(-1)^k F_{ak+d} = (-1)^{n+a}F_{an+d} + (-1)^n F_{an+a+d} - F_{a+d} + (L_a+1)F_d \quad (13)$$

For $d = 0$, (12) yields

$$(1-L_a x + (-1)^a x^2)\sum_{k=0}^{\infty} F_{ak}x^k = F_a x, \quad (a = 0, 1, \ldots) \quad (14)$$

Again, let $q = 1$ and $p = -1$ in (1). Then, for $u_0 = 2$, and $u_1 = 1$, we now have $u_n \equiv L_n$, $S_n \equiv L_n$, and $L_{-n} = (-1)^n L_n$. Thus, with $u_n \equiv L_n$, (5) and (6) become, respectively,

$$(1-L_a x+(-1)^a x^2)\sum_{k=0}^{n} L_{ak+d}x^k = (-1)^a x^{n+2}L_{an+d} - x^{n+1}L_{an+a+d} + xL_{a+d} + (1-xL_a)L_d, \quad (15)$$

$$(1-L_a x + (-1)^a x^2)\sum_{k=0}^{\infty} L_{ak+d}x^k = L_d + (L_{a+d} - L_a L_d)x \quad (16)$$

REFERENCES

1. Ken Siler, Fibonacci summations, Fibonacci Quarterly, Vol. 1, No. 3, October 1963, pp. 67-69.

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