ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-41 *Proposed by Robert A. Laird, New Orleans, La.*

Find rational integers, x, and positive integers, m, so that

$$N = x^2 - m \quad \text{and} \quad M = x^2 + m$$

are rational squares. There are no solutions for $m = 1, 2, 3, 4$ but $m = 5$ is historically interesting.

H-42 *Proposed by J.D.E. Konhauser, State College, Pa.*

A set of nine integers having the property that no two pairs have the same sum is the set consisting of the nine consecutive Fibonacci numbers, 1, 2, 3, 5, 8, 13, 21, 34, 55 with total sum 142. Starting with 1, and annexing at each step the smallest positive integer which produces a set with the stated property yields the set 1, 2, 3, 5, 8, 13, 21, 30, 39 with sum 122. Is this the best result? Can a set with lower total sum be found?

H-43 *Proposed by H.W. Gould, West Virginia University, Morgantown, West Va.*

Let

$$\varphi(x) = \sum_{n=1}^{\infty} x^{F_{mn}} ,$$

where F_j is the j-th Fibonacci number, find

$$\lim_{x \to 1} \frac{(x)}{-\log(1-x)} .$$

See special case $m = 2$ in Revista Matematica Hispano-Americana (2)

9 (1934) 223-225 problem 115.

H-44 *Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Let $u_o = q$ and $u_1 = p$, and $u_{n+2} = u_{n+1} + u_n$, then the u_n are called generalized Fibonacci numbers.

(1) Show $$u_n = pF_n + qF_{n-1}$$

(2) Show that if

$$V_{2n+1} = u_n^2 + u_{n+1}^2 \quad \text{and} \quad V_{2n} = u_{n+1}^2 - u_{n-1}^2 ,$$

then V_n are also generalized Fibonacci numbers.

H-45 *Proposed by R.L. Graham, Bell Telephone Labs., Murray Hill, N.J.*

Prove

$$\sum_{p=0}^{n} \sum_{q=0}^{p} \sum_{r=0}^{q} \sum_{s=0}^{r} F_s^2 = F_{n+2}^2 - \frac{1}{8}(2n^2+8n+11-3(-1)^n) ,$$

where F_n is the nth Fibonacci number.

SOLUTIONS

WARD'S LAST THEOREM

H-24 *Proposed by the late Morgan Ward, California Institute of Technology, Pasadena, California*

Let $\phi_n(x) = x + x^2/2 + \ldots + x^n/n$, and let $k(x) = k_p(x) = (x^{p-1}-1)/p$, where p is an odd prime greater than 5. (The function $k(x)$ is called the "quotient of Fermat" in the literature.) Let $P = P_p$ be the rank of apparition of p in the sequence $0, 1, 1, 2, 3, 5, \ldots, F_n$, (so $P_{13} = 7$, $P_7 = 8$ and so on).

Then

$$F_P \equiv 0 \bmod p^2$$

if and only if

$$\phi_{(p-1)/2}(5/9) = 2k(3/2) \bmod p .$$

Solution by L. Carlitz, Duke University, Durham, N.C.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} .$$

Then the condition $F_P \equiv 0 \pmod{p}$, $p > 5$, is equivalent to $\alpha^P = \beta^P \pmod{p}$, or what is the same thing

$$(*) \qquad (-\alpha^2)^P = 1 \pmod{p} .$$

We now treat separately the two cases

$$\text{(i)} \quad \left(\frac{5}{p}\right) = +1, \qquad \text{(ii)} \quad \left(\frac{5}{p}\right) = -1 ,$$

where $(5/p)$ is the Legendre symbol.

In the first case we have $5 = \pi\pi'$, where π, π' are primes of the quadratic field $R(\sqrt{5})$. It follows that $P \mid p-1$. Then clearly

$$(-\alpha^2)^P = 1 \pmod{p^2} \rightleftarrows (-\alpha^2)^{p-1} = 1 \pmod{p^2} .$$

We therefore consider $\alpha^{2p} - \alpha^2 \pmod{p^2}$. Since

$$\alpha^{2p} = \alpha^2 \pmod{p^2} \rightleftarrows \beta^{2p} = \beta^2 \pmod{p^2} ,$$

it will suffice to consider

$$\alpha^{2p} + \beta^{2p} - 3 \pmod{p^2} .$$

Now

$$\alpha^{2p} + \beta^{2p} = \frac{2}{2^p} \sum_{r=0}^{\frac{1}{2}(p-1)} \binom{p}{2r} 3^{p-2r} 5^r$$

$$= \frac{2}{2^p} \cdot 3^p + \frac{2}{2^p} \sum_{r=1}^{\frac{1}{2}(p-1)} \binom{p}{2r} 3^{p-2r} 5^r .$$

Since

$$\binom{p}{2r} \equiv -\frac{p}{2r} \pmod{p^2} ,$$

we get

$$\alpha^{2p} + \beta^{2p} = 2\left(\frac{3}{2}\right)^p - p\left(\frac{3}{2}\right)^p \sum_{r=1}^{\frac{1}{2}(p-1)} \frac{1}{r} \left(\frac{5}{9}\right)^r$$

$$\equiv 2\left(\frac{3}{2}\right)^p - \frac{3p}{2} \phi_{\frac{1}{2}(p-1)} \left(\frac{5}{9}\right) \pmod{p^2} .$$

Therefore $\alpha^{2p} + \beta^{2p} \equiv 3 \pmod{p^2}$ is equivalent to

$$\frac{3}{2}\phi_{\frac{1}{2}(p-1)}\left(\frac{5}{9}\right) \equiv 2\left(\frac{3}{2}\right)^p - 3$$

$$\equiv 3\left[\left(\frac{3}{2}\right)^{p-1} - 1\right]$$

$$\equiv 3k\left(\frac{3}{2}\right) \pmod{p} \quad .$$

This completes the proof in case (i).

In case (ii) we have $\alpha^p \equiv \beta \pmod{p}$ so that $\alpha^{p+1} \equiv -1$. This implies

$$(-\alpha^2)^{p+1} = 1 \pmod{p} \quad .$$

Since $(-\alpha^2)^{p-1} \not\equiv 1$, comparison with (*) shows that $P \mid p+1$, $P \nmid (p-1)$. Thus we may consider $\alpha^{2p} - \beta^2 \pmod{p^2}$. Since

$$\alpha^{2p} = \beta^2 \pmod{p^2} \rightleftarrows \beta^{2p} = \alpha^2 \pmod{p^2}$$

it suffices to consider

$$\alpha^{2p} + \beta^{2p} - \alpha^2 - \beta^2 \pmod{p^2} \quad .$$

Hence the proof is completed as in case (i).

Also solved by John Halton whose solution will appear in a paper to be published later in the Fibonacci Quarterly Journal

CORRECTED PROBLEM AND SOLUTION

H-25 *Proposed by Joseph Erbacker and John A. Fuchs, University of Santa Clara, and F.D. Parker, Suny, Buffalo, N.Y.*

Prove:

$$D_n = |a_{ij}| = 36, \quad \text{for all } n \ ,$$

where

$$a_{ij} = F^3_{n+i+j-2} \quad (i, j = 1, 2, 3, 4) \quad .$$

Solution by C.R. Wall, Ft. Worth, Texas

Since $\quad F^3_{n+4} - 3F^3_{n+3} - 6F^3_{n+2} + 3F^3_{n+1} + F^3_n = 0 \ ,$

it follows that D_{n+1} can be obtained from D_n by column operations that leave the determinant invariant. Thus $D_n = D_{n+1}$. To evaluate set $n = 0$.

Also solved by the proposers

NO SOLUTIONS

H-26 *Proposed by Leonard Carlitz, Duke University, Durham, N.C.*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-s}$, then show $R_k^n = (a_{rs})$ such that

$$a_{rs} = \sum_{j=0}^{s-1} \binom{r-1}{j}\binom{k+1-r}{s-1-j} F_{n-1}^{k+2-r-s+j} F_n^{r+s-2-2j} F_{n+1}^{j} .$$

GENERATING FUNCTIONS AND CONVOLUTION

H-27 *Proposed by Harlan L. Umansky, Emerson High School, Union City, N.J.*

Show that

$$F_k^3 = \sum_{j=1}^{k-2} (-1)^{j+1} F_j F_{3k-3j} + (-1)^k F_{k-3}, \qquad k \geq 4 .$$

Solution by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

The generating functions below are easily verified

$$\frac{x-2x^2-x^3}{1-3x-6x^2+3x^3+x^4} = \sum_{n=0}^{\infty} F_n^3 x^n$$

$$\frac{2x}{1-4x-x^2} = \sum_{n=0}^{\infty} F_{3n} x^n$$

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} F_n x^n$$

$$\frac{x}{1+x-x^2} = \sum_{n=0}^{\infty} F_n (-1)^{n+1} x^n$$

$$\frac{x-2x^2-x^3}{1-3x-6x^2+3x^3+x^4} = \left[\frac{2x}{(1-4x-x^2)}\right]\left[\frac{x}{1+x-x^2}\right] + \frac{x}{1+x-x^2}$$

Thus

$$\sum_{k=0}^{\infty} F_k^3 x^k = \left(\sum_{k=0}^{\infty} F_{3k} x^k\right)\left(\sum_{k=0}^{\infty} F_k(-1)^{k+1} x^k\right) + \sum_{k=0}^{\infty} F_k(-1)^{k+1} x^k$$

so that, using Cauchy product of series, and equating coefficients, one obtains

$$F_k^3 = \sum_{i=0}^{k} (-1)^{i+1} F_i F_{3k-3i} + (-1)^{k+1} F_k$$

$$= \sum_{i=1}^{k-2} (-1)^{i+1} F_i F_{3k-3i} + (-1)^k \{2F_{k-1} - F_k\}$$

$$= \sum_{i=1}^{k-2} (-1)^{i+1} F_i F_{3k-3i} + (-1)^k F_{k-3} .$$

Also solved by C.R. Wall and the proposer.

A FIBONACCI BEAUTY

H-28 *Proposed by H.W. Gould, West Virginia University, Morgantown, W. Va.*

Let $C_j(r, n)$ be the number of numbers, to the base $r (r \geq 2)$ with at most n digits, and the sum of the digits equal to j.

Sum the series:

$$\sum_{j=0}^{\infty} C_j(r, n) a^j b^{rn-n-j} .$$

Solution by John H. Halton, University of Colorado, Boulder, Colorado, and Leonard Carlitz, Duke University, Durham, N.C.

Let

$$S_n(r, a, b) = \sum_{j=0}^{\infty} C_j(r, n)\, a^j b^{rn-n-j} = b^{(r-1)n} \sum_{n=0}^{r^{n-1}} \left(\frac{a}{b}\right)^{N_o+N_1+\ldots+N_{n-1}}$$

where

$$N = N_o + N_1 r + N_2 r^2 + \ldots + N_{n-1} r^{n-1}, \quad 0 \leq N_i \leq r-1 .$$

Then

$$S_n = b^{(r-1)n} \sum_{N_o=0}^{r-1} \sum_{N_1=0}^{r-1} \cdots \sum_{N_{n-1}=0}^{r-1} \left(\frac{a}{b}\right)^{N_o+N_1+\ldots N_{n-1}}$$

$$= \prod_{m=0}^{n-1} \left\{ b^{r-1} \sum_{N_m=0}^{r-1} \left(\frac{a}{b}\right)^{N_m} \right\} = \prod_{m=1}^{n} \left\{ \frac{a^r-b^r}{a-b} \right\} = \left(\frac{a^r-b^r}{a-b}\right)^n$$

If

$$a = \frac{1}{2}(1+\sqrt{5}),\ b = \frac{1}{2}(1-\sqrt{5}),\ \text{then}\ \frac{a^r-b^r}{a-b} = F_r\ ,$$

the Fibonacci number. In that case,

$$S_n\left(r,\ \frac{1}{2}(1+\sqrt{5}),\ \frac{1}{2}(1-\sqrt{5})\right) = F_r^n\ .$$

Also solved by the proposer.

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A DIGIT MUSES* ....

Oh!
4
2B
No zero
In the world of math!
Would that I were like that great
Built into the structure of the universe and art
The ideal of ideals dividing all things in proportions of gold — a paragon!

Brother U. Alfred

*This poem has the distinction that the number of syllables in each line proceeds by the sequence: 1, 1, 2, 3, 5, 8, 13, 21.

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