A PARTIAL DIFFERENCE EQUATION RELATED TO THE FIBONACCI NUMBERS

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1. Consider the equation

$$u_{mn} - u_{m-1,n} - u_{m,n-1} - u_{m-2,n} + 3u_{m-1,n-1} - u_{m,n-2} = 0 \qquad (m \geq 2, \; n \geq 2). \tag{1.1}$$

If we put

$$G(x, y) = \sum_{m,n=0}^{\infty} u_{mn} x^m y^n \tag{1.2}$$

and

$$f(x, y) = 1 - x - y - x^2 + 3xy - y^2 , \tag{1.3}$$

it follows from (1.1) that

$$f(x, y)G(x, y) = a + bx + cy , \tag{1.4}$$

where a, b, c are constants. Indeed it is evident that

$$a = u_{00}, \quad b = u_{10} - u_{00}, \quad c = u_{01} - u_{00} . \tag{1.5}$$

Thus if u_{00}, u_{10}, u_{01}, or equivalently a, b, c, are assigned u_{mn} is uniquely determined for all non-negative integers m, n. We shall show that the general solution of (1.1) can be expressed in terms of Fibonacci numbers.

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2. If we put

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \qquad \beta = \frac{1}{2}(1 - \sqrt{5}), \tag{2.1}$$

it is easily verified that

$$(1 - \alpha x - \beta y)(1 - \beta x - \alpha y) = 1 - (\alpha + \beta)(x + y) + \alpha\beta\,(x^2 + y^2) + (\alpha^2 + \beta^2)xy$$

$$= 1 - x - y - x^2 + 3xy - y^2 ,$$

so that

$$f(x, y) = (1 - \alpha x - \beta y)(1 - \beta x - \alpha y) . \tag{2.2}$$

We now consider the case

$$a = 0, \quad b = 1, \quad c = -1 . \tag{2.3}$$

Then

$$\frac{x-y}{f(x, y)} = \frac{1}{\alpha - \beta}\left[\frac{1}{1 - \alpha x - \beta y} - \frac{1}{1 - \beta x - \alpha y}\right]$$

$$= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left\{(\alpha x + \beta y)^n - (\beta x + \alpha y)^n\right\}$$

$$= \frac{1}{\alpha - \beta} \sum_{m,\, n=0}^{\infty} \binom{m+n}{n}(\alpha^m \beta^n - \alpha^n \beta^m)\, x^m y^n .$$

If F_{mn} denotes the solution of (1.1) and (2.3) holds, we have therefore

$$F_{mn} = \binom{m+n}{m} \frac{\alpha^m \beta^n - \alpha^n \beta^m}{\alpha - \beta} \tag{2.4}$$

Now it is evident from (1.4) and (2.3) that

$$F_{mn} = -F_{nm}, \quad F_{nn} = 0, \tag{2.5}$$

so that it will suffice to determine F_{mn} when $m > n$.

If as usual we put

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.6}$$

then it follows from (2.4) that

$$F_{mn} = (-1)^n \binom{m+n}{m} F_{m-n} \quad (m \geq n) . \tag{2.7}$$

In view of (2.5), this result can be expressed in the following form:

$$\frac{x-y}{f(x,y)} = \sum_{m > n} (-1)^n \binom{m+n}{n} F_{m-n} \left(x^m y^n - x^n y^m\right) . \tag{2.8}$$

We can also evaluate

$$\Phi(x,y) = \sum_{m=1}^{\infty} \sum_{n=0}^{m-1} F_{mn} x^m y^n . \tag{2.9}$$

Indeed, by (2.7), we have

$$\Phi(x,y) = \sum_{n=0}^{\infty} (-1)^n x^n y^n \sum_{k=0}^{\infty} \binom{k+2n}{n} F_k x^k$$

$$= \sum_{k=0}^{\infty} F_k x^k \sum_{n=0}^{\infty} (-1)^n \binom{k+2n}{n} x^n y^n .$$

Now it is known that

$$\sum_{n=0}^{\infty} \binom{k+2n}{n} x^n = \frac{1}{\sqrt{1-4x}} \left(\frac{2}{1+\sqrt{1-4x}} \right)^k ,$$

so that

$$\Phi(x, y) = \frac{1}{\sqrt{1+4xy}} \sum_{k=0}^{\infty} F_k \left(\frac{2x}{1+\sqrt{1-4x}} \right)^k$$

This reduces to

$$\Phi(x, y) = \frac{z}{\sqrt{1+4xy}(1-z-z^2)}, \quad z = \frac{2x}{1+\sqrt{1-4xy}} . \tag{2.10}$$

We have also

$$\Phi(x, y) - \Phi(y,x) = \frac{x-y}{f(x, y)} . \tag{2.11}$$

It is not difficult to verify that

$$\frac{1}{1-\alpha z} = (1+\sqrt{1+4xy}) \frac{1-\sqrt{1+4xy}-2\alpha x}{-4\alpha x(1-\alpha x-\beta y)} ,$$

so that

$$\frac{z}{1-z-z^2} = (1+\sqrt{1-4xy}) \frac{-1+x+2x^2-2xy+(1-x)\sqrt{1-4xy}}{4xf(x, y)}$$

$$= \frac{x+y-2xy+(x-y)\sqrt{1-4xy}}{2f(x, y)} .$$

It follows that

$$\Phi(x, y) - \Phi(y, x) = \frac{2(x-y)}{2f(x, y)} = \frac{x-y}{f(x, y)} ,$$

in agreement with (2.11).

3. We next take the case

$$a = 2, \quad b = c = -1 . \tag{3.1}$$

Then

$$\frac{2-x-y}{f(x,y)} = \frac{1}{1-\alpha x-\beta y} + \frac{1}{1-\beta x-\alpha y} = \sum_{n=0}^{\infty} \left\{ (\alpha x+\beta y)^n + (\beta x+\alpha y)^n \right\}$$

$$= \sum_{m,n=0}^{\infty} \binom{m+n}{m} (\alpha^m \beta^n + \alpha^n \beta^m)\, x^m y^n .$$

Thus, if L_{mn} denotes the solution of (1.1) when (3.1) holds, we have

$$L_{mn} = \binom{m+n}{m} (\alpha^m \beta^n + \alpha^n \beta^m) . \tag{3.2}$$

Also it is evident from (1.4) and (3.1) that

$$L_{mn} = L_{nm} , \tag{3.3}$$

so it will suffice to evaluate L_{mn} when $m \geq n$. If we put

$$L_n = \alpha^n + \beta^n \tag{3.4}$$

it follows from (3.2) that

$$L_{mn} = (-1)^n \binom{m+n}{m} L_{m-n} \qquad (m \geq n) . \tag{3.5}$$

By (3.3) this result can be stated in the form

$$\frac{2-x-y}{f(x,y)} = 2 \sum_{n=0}^{\infty} (-1)^n \binom{2n}{n} x^n y^n \sum_{m>n} (-1)^n \binom{m+n}{m} L_{m-n} (x^m y^n + x^n y^m). \tag{3.6}$$

4. We now take

$$a = 1, \quad b = c = 0 \tag{4.1}$$

and let G_{mn} denote the solution of (1.1) in this case. Thus it is clear that

$$\frac{1}{f(x, y)} = \sum_{m,n=0}^{\infty} G_{mn} x^m y^n . \tag{4.2}$$

Comparing this with

$$\frac{x-y}{f(x, y)} = \sum_{m,n=0}^{\infty} F_{mn} x^m y^n$$

we get

$$(x-y) \sum_{m,n=0}^{\infty} G_{mn} x^m y^n = \sum_{m,n=0}^{\infty} F_{mn} x^m y^n ,$$

so that

$$G_{m-1,n} - G_{m,n-1} = F_{mn} \quad (m \geq 1, \; n \geq 1) . \tag{4.3}$$

It is evident from (4.2) that

$$G_{mn} = G_{nm} \tag{4.4}$$

and

$$G_{mo} = G_{om} = F_{m+1} . \tag{4.5}$$

If $m \geq n$ it follows from (4.3) and (2.7) that

$$G_{mn} = F_{m+1,\,n+1} + (-1)^n F_{m-n+1} .$$

Repeated application of this formula leads to

$$(4.6) \qquad G_{mn} = \sum_{r=0}^{n} (-1)^r \binom{m+n+1}{r} F_{m+n-2r+1} \qquad (m \geq n) .$$

By (4.4) this result can be stated in the following form,

$$(4.7) \qquad \begin{aligned} \frac{1}{f(x,y)} &= \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{2n+1}{r} F_{2n-2r+1}\, x^n y^n \\ &+ \sum_{m>n} \sum_{r=0}^{n} (-1)^r \binom{m+n+1}{r} F_{m+n-2r+1} (x^m y^n + x^n y^m) . \end{aligned}$$

5. It is now easy to express the general solution of (1.1) in terms of F_{mn}, L_{mn}, G_{mn} and therefore in terms of F_k and L_k. As we have seen above, if the numbers u_{00}, u_{10}, u_{01} are assigned, u_{mn} is uniquely determined for all $m, n \geq 0$. Indeed we may put

$$(5.1) \qquad u_{mn} = AF_{mn} + BL_{mn} + CG_{mn} ,$$

where A, B, C are independent of m, n. Then

$$(5.2) \qquad \begin{cases} u_{00} = AF_{00} + BL_{00} + CG_{00} \\ u_{10} = AF_{10} + BL_{10} + CG_{10} \\ u_{01} = AF_{01} + BL_{01} + CG_{01} . \end{cases}$$

But by (2.7), (3.5), (4.4) and (4.5)

$$F_{00} = 0, \quad L_{00} = 2, \quad G_{00} = 1$$
$$F_{10} = 1, \quad L_{10} = 1, \quad G_{10} = 1$$
$$F_{01} = -1, \quad L_{01} = 1, \quad G_{01} = 1 \quad .$$

Substituting these values in (5.2) we find that

$$(5.3) \qquad \begin{cases} A = \frac{1}{2}(u_{10} - u_{01}) \\ B = u_{00} - \frac{1}{2}(u_{10} + u_{01}) \\ C = -u_{00} + u_{10} + u_{01} \quad . \end{cases}$$

Thus (5.1) becomes

$$(5.4) \qquad \begin{aligned} u_{mn} = \frac{1}{2}(u_{10} - u_{01})F_{mn} &+ (u_{00} - \frac{1}{2}u_{10} - \frac{1}{2}u_{01})L_{mn} \\ &+ (-u_{00} + u_{10} + u_{01})G_{mn} \quad . \end{aligned}$$

Finally, making use of (2.7), (3.5) and (4.6), we can express u_{mn} explicitly in terms of F_k and G_k.

6. It is of some interest to extend the solutions of (1.1) to arbitrary integral values of m and n. In the first place we define F_{mn} by means of

$$(6.1) \qquad F_{mn} = \binom{m+n}{m} \frac{\alpha^m \beta^n - \alpha^n \beta^m}{\alpha - \beta}$$

for all integral m, n. Now since

$$\binom{m-n}{m} = 0 \qquad (o < n \le m)$$

it follows that

(6.2) $$F_{m,-n} = 0 \quad (o < n \leq m) ;$$

similarly we have

(6.3) $$F_{-m,n} = 0 \quad (o < m \leq n) .$$

Also since, by definition,

$$\binom{-m-n}{-m} = 0 \quad (m > o,\ n > o)$$

we have

(6.4) $$F_{-m,-n} = 0 \quad (m > o,\ n > o) .$$

On the other hand, since

$$\binom{m-n}{m} = (-1)^m \binom{n-1}{m} \quad (n > m) ,$$

it follows that

(6.5) $$F_{m,-n} = (-1)^{m+n} \binom{n-1}{m} F_{m+n} \quad (n > m) ;$$

similarly

(6.6) $$F_{-m,n} = -(-1)^{m+n} \binom{m-1}{n} F_{m+n} \quad (m > n) .$$

Note that in all cases we have

(6.7) $$F_{mn} = - F_{nm} .$$

We remark that if we define

(6.8) $$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for all integral n, then (6.1) becomes

(6.9) $$F_{mn} = (-1)^n \binom{m+n}{m} F_{m-n} = -(-1)^m \binom{m+n}{m} F_{n-m}.$$

It remains to show that F_{mn} as defined by (6.1) or (6.9) does satisfy (1.1) for all m, n. We have

$$\begin{aligned} &F_{mn} - F_{m-1,n} - F_{m,n-1} - F_{m-2,n} + 3F_{m-1,n-1} - F_{m,n-2} \\ &= (-1)^n \binom{m+n}{m} F_{m-n} - (-1)^n \binom{m+n-1}{m-1} F_{m-n-1} \\ &\quad + (-1)^n \binom{m+n-1}{m} F_{m-n+1} - (-1)^n \binom{m+n-2}{m-2} F_{m-n-2} \\ &\quad - 3(-1)^n \binom{m+n-2}{m-1} F_{m-n} - (-1)^n \binom{m+n-2}{m} F_{m-n+2}. \end{aligned}$$

Now making use of

$$F_{n+1} = F_n + F_{n-1},$$

which holds for all integral n, we find that F_{mn} satisfies (1.1).

The extension of L_{mn} can be carried out in exactly the same way. We define

(6.10) $$L_{mn} = (-1)^n \binom{m+n}{m} L_{m-n}$$

for all integral m, n, where

(6.11) $$L_n = \alpha^n + \beta^n$$

for all integral n.

As for G_{mn}, we require that

(6.12) $$G_{m-1,n} - G_{m,n-1} = F_{mn}$$

for all m, n. If n is negative we replace n by $-n$, so that (6.12) becomes

$$G_{m-1,-n} - G_{m,-n-1} = F_{m,-n} .$$

This may be written as

$$G_{m,-n-1} = G_{m-1,-n} - F_{m,-n} ,$$

which implies

$$G_{m,-n} = G_{m-n,o} - \sum_{r=o}^{n-1} F_{m-r,-n+r-1} .$$

We put (compare (4.5))

$$G_{m0} = G_{0m} = F_{m+1} \tag{6.13}$$

for all m; it follows that

$$G_{m,-n} = F_{m-n+1} - \sum_{r=o}^{n-1} F_{m-r,-n+r+1} \quad (n \geq 1) . \tag{6.14}$$

Similarly if m is negative we get

$$G_{-m,n} = F_{n-m+1} + \sum_{r=o}^{m-1} F_{r-m+1,n-r} \quad (m \geq 1) . \tag{6.15}$$

Indeed we find that $G_{-m,n}$ as defined by (6.15) satisfies (6.12) for all n. It can be verified easily that

$$G_{mn} = G_{nm} \tag{6.16}$$

for all m, n.

Finally we can show that G_{mn} as defined by (4.6), (6.14) and (6.15) satisfies (1.1). We omit the details of this verification.

7. We remark that the difference equation (1.1) can be generalized in an obvious way. Let α, β be roots of the quadratic equation

$$x^2 - px + q = 0, \tag{7.1}$$

where p, q are arbitrary numbers, and put

$$f(x, y) = (1 - \alpha x - \beta y)(1 - \beta x - \alpha y) = 1 - p(x+q) + qx^2 + (p^2 - 2q)xy + qy^2 .$$

Then the generalized equation is

$$\begin{aligned} u_{m,n} - pu_{m-1,n} - pu_{m,n-1} + qu_{m-2,n} \\ + (p^2 - 2q)u_{m-1,n-1} + qu_{m,n-2} = 0 . \end{aligned} \tag{7.2}$$

The results obtained above for (1.1) can be carried over without difficulty to the more general equation (7.2).

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Continued from page 176.

Equating coefficients in (1) and (3), one obtains, the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} .$$

If, on the other hand we let y = 2, y' = 1; x = 0, equation (1) becomes

$$y = e^{\alpha x} + e^{\beta x} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n) \frac{x^n}{n!} .$$

The series solution yields $u_0 = 2$ and $u_1 = 1$ so that equation (3) becomes

$$y = \sum_{n=0}^{\infty} \frac{L_n x^n}{n!} ,$$

and one obtains

$$L_n = \alpha^n + \beta^n .$$

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