FURTHER COMMENTS ON THE PERIODICITY OF THE DIGITS OF THE FIBONACCI SEQUENCE

RICHARD L. HEIMER
Airborne Instrument Laboratory, Deer Park, New York

In the Fibonacci Quarterly, Volume 1, Number 4, Dov Jarden showed that the last $d \geq 3$ digits of the Fibonacci numbers repeat every $15 \cdot 10^{d-1}$ times. He also commented on Stephen P. Geller's announcement of the periodicity of the first three digits. Summarizing these results, we find that the digits of the Fibonacci series, 0, 1, 1, 2, 3, 5 .... repeat as follows:

Table I

	Units 1	Tens 2	3	4	5		d
Repetition Period	60	300	1500	1.5×10^4	1.5×10^5	.....	1.5×10^d

What aroused my interest in the sequence of repetition periods was that the repetition periods of the tens and hundreds digits were five times the value of the repetition periods of the units and tens digit respectively, while the remaining repetition periods increased by a factor of ten. Also I was interested in possible discovering any governing relationships which caused 60 to be the periodicity of the units digit as well as a factor of subsequent periods. Believing that number systems with different bases (i. e., Binary, Ternary, etc.) would also display these periodic qualities, I proceeded to generate Fibonacci series in these bases, and in the course of so doing, discovered further significant periodic properties of these sequences.

The units and tens digits were generated from Base 2 thru Base 16 while the third and fourth (hundreds and thousands) were derived only where it was practical to do so. These results are shown in Table II which tabulates the periodicity of the digits in the various bases as well as the corresponding multiplying factors which show the relationship between the repetition periods of the digits in a given base.

Table II

BASE	REPETITION PERIOD OF DIGIT 1	2	3	4	MULTIPLYING FACTOR $\frac{R_{D_2}}{R_{D_1}}$	$\frac{R_{D_3}}{R_{D_2}}$	$\frac{R_{D_4}}{R_{D_3}}$
2	3	6	12	24	2	2	2
3	8	24	72		3	3	
4	6	24	96		4	4	
5	20	100			5		
6	24	144			6		
7	16	112			7		
8	12	96			8		
9	24	216			9		
10	60	300	1500	15,000	5	5	10
11	10	110			11		
12	24	24	288		1	12	
13	28	364			13		
14	48	336			7		
15	40	600			15		
16	24	96			4		

The first conclusion which may be drawn from this table is that the multiplying factors are either the number base or a rational fraction thereof and that:

(1) $$R_D = A_B \cdot F \cdot B^D$$

Where: R_D is the periodicity of the $D^{\underline{th}}$ digit.

A_B is the repetition period of the units digit divided by the base B.

F is a rational fraction.

A further observation reveals that for a given digit:

> The product of two bases is a third base whose periodicity is the product of the repetition periods of the original two bases or a rational fraction thereof.

Formulating this, we have for a given digit:

(2) $$R_{B_x} \cdot R_{B_y} = KR_{B_{xy}}$$

Where K is a rational fraction. As an example of this note that for units digit:

(3) $$R_{B_2} = 3, \; R_{B_3} = 8, \qquad R_{B_2} \cdot R_{B_3} = 3 \cdot 8 = 24 = R_{B_{2\cdot 3}}$$

It also may be seen that in the decimal system:

$$R_{B_{10}} = 60 = R_{B_{5\cdot 2}} = R_{B_5} \times R_{B_2} = 20 \cdot 3 = 60$$

This of course only partially satisfies my understanding of the nature of the decimal base units digit repeating every 60 Fibonacci numbers.

While generating these sequences another significant result occurred which is the relationship of the digits within a given repetition cycle. This relationship is best indicated by way of illustration. Regard the units, tens, and hundreds digits of the Fibonacci sequence in the base 3: [(000, 001, 001, 002, 010, 012, 022, 111) (210, 021, 001, 022, 100, 122, 222, 121) (120, 011, 201, 212, 120, 102, 222, 101)] [(100, 201 ......). The parentheses are drawn around the repeating units digit which occur at intervals of 8. The brackets are drawn after an interval of 24.

Note what happens if the tens digit only is written in 3 horizontal groups of 8 (8 being the periodicity of the digit preceding the tens digit, the units digit).

	0	0	0	0	1	1	2	1
	1	2	0	2	0	2	2	2
	2	1	0	1	2	0	2	0
Difference:	1	2	0	2	2	1	0	1

The difference between neighboring vertical terms is indicated next to the heading of "DIFFERENCE." Note that the differences are a Fibonacci type sequence where any term after the first two is the sum of the previous two in that base.

This relationship has held for all the sequences in all the bases that I have investigated. Assuming that it holds generally, the remaining hundreds digit sequence of the base 3 example may be generated as follows:

Step 1: Calculate the digit desired (hundreds) for one period (24) of the preceding digit (tens). Now calculate two more terms (25 and 26). List the first group horizontally and the two subsequent terms on the 2nd line under the first two terms of the first line. (See Illustration I.)

Step 2: On top of this calculate the difference of the first two sets of vertical terms (see Illustration I).

Step 3: Beginning with these two differences generate a Fibonacci sequence in that base (see Illustration I).

Step 4: Using the generated differences, vertically fill in the remaining terms of the digital sequence as follows:

Illustration I

Step 3

Differences: Step 2 →	1	2	0	2	2	1	0	1	1	2	0	2	2	1	0	1	1	2	0	2	2	1	0	1
Step 1 →	0	0	0	0	0	0	0	1	2	0	0	0	1	1	2	1	1	0	2	2	1	1	2	1
	1	2	0	2	2	1	0	2	0	2	0	2	0	2	2	2	2	2	2	1	0	2	2	2
	2	1	0	1	1	2	0	0	1	1	0	1	2	0	2	0	0	1	2	0	2	0	2	0

Step 4

Between Step 1 and Step 4 we have completely specified one period of the hundreds digit without making 72 three digit additions which this would normally require. This method also provides quick prediction of fractional multiplying factors mentioned earlier.

I am reporting these comments in hope that someone may further develop these thoughts in this fascinating sub-sub-field of Fibonacci sequences.

XXXXXXXXXXXXXX