LINEAR RECURRENCE RELATIONS - PART III

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1. INTRODUCTION

We continue our study to acquaint the beginner with linear recurrence relations and the method of generating functions for solving them (see articles in [1] and [2]). In this concluding article we shall consider recurrence relations in which there is more than one independent variable.

2. DEFINITION

A partial linear recurrence relation in two independent variables m, n is as equation of the form

$$\sum_{i=0}^{k} \sum_{j=0}^{p} a_{ij}(m, n)\, y(m+i, n+j) = b(m, n) \tag{2.1}$$

where a_{ij} and b are given functions of the discrete variables m and n over the set of non-negative integers. Partial recurrence relations in three or more independent variables may be defined in a similar way.

If $b(m, n) = 0$, relation (2.1) is called homogeneous. The equation contains $(k+1)(p+1)$ possible terms and is said to be of order k with respect to m and of order p with respect to n. To solve certain recurrence relations we find it convenient to apply a generating function transform.

3. A SERIES TRANSFORM

The exponential generating function for the sequence $\{y(m, n)\}$, $(m, n = 0, 1, 2, \ldots)$ is defined by the double infinite series

$$Y(s, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m, n) \frac{s^m}{m!} \frac{t^n}{n!} \tag{3.1}$$

If the series (3.1) converges when $|s| < \alpha$ and $|t| < \beta$ simultaneously, then all of the derived series of (3.1) will also converge in the same region. Thus, we have

$$\frac{\partial Y}{\partial s} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} m\, y(m,n) \frac{s^{m-1}}{m!} \frac{t^n}{n!}$$

(3.2)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m+1,n) \frac{s^m}{m!} \frac{t^n}{n!}$$

and it is seen that $(\partial Y/\partial s)$ is the exponential generating function of the sequence $y(m+1,n)$. Similarly, one easily obtains the equations

$$\frac{\partial Y}{\partial t} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m,n+1) \frac{s^m}{m!} \frac{t^n}{n!} \tag{3.3}$$

and

$$\frac{\partial^2 Y}{\partial s \partial t} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m+1,n+1) \frac{s^m}{m!} \frac{t^n}{n!}, \tag{3.4}$$

which furnish exponential generating functions for the sequences $\{y(m,n+1)\}$ and $\{y(m+1,n+1)\}$ respectively. In general, the relation

$$\frac{\partial^{i+j} Y}{\partial s^i \partial t^j} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y(m+i,n+j) \frac{s^m}{m!} \frac{t^n}{n!} \tag{3.5}$$

is the exponential generating function of the sequence $\{y(m+i,n+j)\}$. This equation permits us to transform linear partial recurrence relations (2.1) where the coefficients $a_{ij}(m,n)$ are all assumed to be constants (i.e., not a function of m and n). If we then multiply both sides of (2.1) by $\frac{s^m}{m!}\frac{t^n}{n!}$ and sum on m and n from zero to infinity, we get the transformed equation

$$\sum_{i=0}^{k} \sum_{j=0}^{p} a_{ij} \frac{\partial^{i+j} Y}{\partial s^i \partial t^j} = B(s,t) , \tag{3.6}$$

where

$$B(s,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b(m,n) \frac{s^m}{m!} \frac{t^n}{n!} . \tag{3.7}$$

After the transformed equation is solved for Y, we then obtain the sequence $\{ y(m,n) \}$ either from the relation

$$y(m,n) = \left. \frac{\partial^{m+n} Y}{\partial s^m \partial t^n} \right]_{\substack{s = 0, \\ t = 0}} \tag{3.8}$$

or by expanding the function $Y(s,t)$ in the form (3.1).

We illustrate the procedure with two simple examples in which $b(m,n) = 0$.

4. EXAMPLES

Consider, for instance, the partial recurrence relation

$$y(m+1,n+1) - y(m+1,n) - y(m,n) = 0 \tag{4.1}$$

with the given conditions

$$\begin{cases} y(m,0) = 0 & \text{if } m \neq 0 \\ y(0,n) = 1 & \\ y(m,n) = 0 & \text{if } m < n . \end{cases} \tag{4.2}$$

The transformed equation for (4.1) is then

$$\frac{\partial^2 Y}{\partial s \partial t} - \frac{\partial Y}{\partial s} - Y = 0 \tag{4.3}$$

with the conditions

4.4) $$Y(s, 0) = 1, \qquad Y(0, t) = e^t .$$

Now, a particular solution of (4.3) is

(4.5) $$Y(s, t) = e^y I_0(2\sqrt{st}) ,$$

where $I_0(z)$ is the modified Bessel function of the first kind defined by

(4.6) $$I_0(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{(m!)^2} .$$

We can obtain the sequence $\{y(m, n)\}$ by expanding (4.5). Thus,

(4.7) $$\begin{aligned} Y(s, t) &= e^t \sum_{m=0}^{\infty} \frac{s^m t^n}{(m!)^2} \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{m=0}^{\infty} \frac{s^m t^n}{(m!)^2} . \end{aligned}$$

Letting $n = m+j$, we then have

(4.8) $$\begin{aligned} Y(s, t) &= \sum_{m=0}^{\infty} \frac{s^m t^m}{(m!)^2} \sum_{n=m}^{\infty} \frac{t^{n-m}}{(n-m)!} \frac{n!}{n!} \\ &= \sum_{m=0}^{\infty} \frac{s^m}{m!} \sum_{n=m}^{\infty} \binom{n}{m} \frac{t^n}{n!} . \end{aligned}$$

Hence, from (3.1) it is clear that

(4.9) $$\begin{aligned} y(m, n) &= \binom{n}{m} , \quad (m = 0, 1, \ldots, n) \\ &= 0 , \quad n < m , \end{aligned}$$

which simply represents the elements of Pascal's triangle (that is, the binomial coefficients).

As a second example, we take the partial recurrence relation

$$(4.10) \qquad y(m+1, n+1) - y(m, n+1) - y(m, n) = 0$$

with the conditions

$$(4.11) \qquad y(0, n) = F_n; \quad y(m, 0) = F_m ,$$

where F_n denotes the nth Fibonacci number. Transformation of equation (4.10) yields

$$(4.12) \qquad \frac{\partial^2 Y}{\partial s\, \partial t} - \frac{\partial Y}{\partial t} - Y = 0$$

with the conditions

$$(4.13) \qquad \begin{aligned} Y(s, 0) &= \frac{1}{\sqrt{5}} \left[e^{sa_1} - e^{sa_2} \right] \\ Y(0, t) &= \frac{1}{\sqrt{5}} \left[e^{ta_1} - e^{ta_2} \right] \end{aligned}$$

where $a_1 = \frac{1}{2}\left(1 + \sqrt{5}\right), \quad a_2 = \frac{1}{2}\left(1 - \sqrt{5}\right)$.

The solution of equation (4.12) is

$$(4.15) \qquad Y(s, t) = \frac{1}{\sqrt{5}} \left[e^{a_1(t+s)} - e^{a_2(t+s)} \right] .$$

Now, employing the inverse transform (3.8) yields

$$(4.16) \qquad y(m, n) = \left. \frac{\partial^{m+n} Y}{\partial s^m \, \partial t^n} \right]_{\substack{s=0 \\ t=0}} = \frac{1}{\sqrt{5}} \left(a_1^{m+n} - a_2^{m+n} \right)$$

which is the solution of (4.10) and represents a Fibonacci array shown in the following table

n \ m	0	1	2	3	4	5	...
0	0	1	1	2	3	5	...
1	1	1	2	3	5	8	
2	1	2	3	5	8	13	
3	2	3	5	8	13	21	
4	3	5	8	13	21	34	
5	5	8	13	21	34	55	
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Fibonacci arrays of higher dimension can also be obtained. These involve the solutions of partial recurrence relations in three or more independent variables.

5. CONCLUDING REMARKS

The above examples involved the solution of two partial recurrence relations having only constant coefficients. Recurrence relations with polynomial coefficients may also be transformed by the method of generating functions. For instance, it is easy to show that the recurrence relation

$$(5.1)\qquad \sum_{i=0}^{k}\sum_{j=0}^{p}(\alpha_{ij}+m\beta_{ij}+n\gamma_{ij})\,y(m+i,n+j) = b(m,n)\ ,$$

having linear coefficients, can be transformed to the equation

$$(5.2)\qquad \sum_{i=0}^{k}\sum_{j=0}^{p}(\alpha_{ij}+\beta_{ij}\phi+\gamma_{ij}\psi)\cdot\frac{\partial^{i+j}Y}{\partial s^{i}\,\partial t^{j}} = B(s,t)\ ,$$

where B(s, t) is given by (3.7), and ϕ and ψ are the differential operators

$$\phi = s\frac{\partial}{\partial s}, \qquad \psi = t\frac{\partial}{\partial t} . \tag{5.3}$$

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REFERENCES

1. J. A. Jeske, "Linear Recurrence Relations — Part I," The Fibonacci Quarterly, Vol. 1, No. 2, pp. 69-74.
2. __________, "Linear Recurrence Relations — Part II," The Fibonacci Quarterly, Vol. 1, No. 4, pp. 35-39.

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ASSOCIATIVITY AND THE GOLDEN SECTION

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E. T. Bell, A functional equation in arithmetic, Trans. Amer. Math. Soc., 39(1936), 341-344, gave a discussion of some matters suggested by the functional equation of associativity

$$\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z) .$$

As a prelude, Bell noted the following theorem.

THEOREM 1. The only polynomial solutions of $\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)$ in the domain of complex numbers are the unsymmetric solutions $\varphi(x, y) = x$, $\varphi(x, y) = y$, and the symmetric solution

$$\varphi(x, y) = a + b(x + y) + cxy ,$$

in which a, b, c, are any constants such that $b^2 - b - ac = 0$.

It is amusing to note a special case. The operation defined by

$$x * y = 1 + b(x + y) + xy$$

is associative only if $b = \frac{1}{2}(\pm\sqrt{5})$.

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