AN ANALYTIC PROOF OF THE FORMULA FOR F_n

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The Fibonacci sequence is defined recursively by the relationship $F_{n+1} = F_n + F_{n-1}$ for $n \geqq 1$, while $F_0 = F_1 = 1$. The two most common procedures for expressing F_n as an explicit function of n are, rather naturally, "finite" in nature. The first of these methods employs the principle of finite induction, while the second involves the solution of a simple finite difference equation. In the present paper I wish to make an analytic attack on the problem employing in particular the theory of residues. The use of such powerful weapons to solve such a simple problem may seem rather absurd, but I am hopeful that the paper may serve as an elementary example of the analytic techniques which have been employed so successfully in attacking very deep and difficult questions in the theory of numbers.

As is well known, the generating function of the Fibonacci numbers is given by

$$f(z) = 1/(1-z-z^2) = \sum_{n=0}^{\infty} F_n z^n .$$

If we consider z to be a complex variable then $f(z)$ is an analytic function whose only singularities are simple poles at the points

$$r = (-1+\sqrt{5})/2 \quad \text{and} \quad s = (-1-\sqrt{5})/2 .$$

r and s, of course, are the roots of the equation $z^2 + z - 1 = 0$. By Cauchy's integral theorem we have

$$F_n = f^{(n)}(0)/n! = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z^{n+1}}$$

where C is the circle $|z| = 1/2$. If Γ is any circle with center at the origin and radius greater than $|s|$ then by Cauchy's residue theorem

(1) $$F_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{z^{n+1}} - (R_1+R_2)$$

where R_1 and R_2 are the residues of $f(z)/z^{n+1}$ at the poles r and s respectively.

Now

$$R_1 = \lim_{z \to r} (z-r)f(z)/z^{n+1} = 1/\left((s-r)r^{n+1}\right),$$

and

$$R_2 = \lim_{z \to s} (z-s)f(z)/z^{n+1} = -1/\left((s-r)s^{n+1}\right).$$

Since $rs = -1$ and $r-s = \sqrt{5}$ we have after simplification

$$(2) \qquad -(R_1+R_2) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

If Γ is the circle $|z| = S$ then, since on Γ $|f(z)| \leqq \dfrac{1}{S^2-S-1}$, we have

$$(3) \qquad \left| \frac{1}{2\pi i}\int_\Gamma \frac{f(z)dz}{z^{n+1}} \right| \leqq \frac{2\pi S}{2\pi S^{n+1}(S^2-S-1)} = \frac{1}{S^n(S^2-S-1)}.$$

Since S may be taken arbitrarily large we conclude from (1), (2), and (3) that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right).$$

Editorial Note: Since $rs = -1$, then

$$r^{-(n+1)} = (-s)^{n+1}$$

and

$$s^{-(n+1)} = (-r)^{n+1},$$

where

$$-s = \frac{1+\sqrt{5}}{2} \text{ and } -r = \frac{1-\sqrt{5}}{2}.$$

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