and if $d > 0$, then

$$\frac{d(d+1)}{2} \le n \Longleftrightarrow d \le \frac{-1+\sqrt{1+8n}}{2} \tag{2}$$

Thus $C_0(n)$ is at least the number of odd divisors d of n that are $\le \sqrt{n}$, so *a fortiori* we have

$$C_0(n) \ge \tau_0(m, \sqrt{m}).$$

If $d|m$ and $d \le \sqrt{m}$, then $m/d|m$ and $m|d \ge \sqrt{m}$. Thus

$$\tau(m, \sqrt{m}) = \begin{cases} \dfrac{\tau(m)}{2} & \text{if } m \text{ is not a square} \\ \dfrac{\tau(m)+1}{2} & \text{if } m \text{ is a square.} \end{cases}$$

Hence $C_0(n) \ge \tau(m)/2$. We have $C_1(n) = \tau(n) - C_0(n)$, and thus

$$C_1(n) \le (k+1)\tau(m) - \frac{\tau(m)}{2} = \left(k + \frac{1}{2}\right)\tau(m).$$

This completes the proof. □

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CONCERNING A PAPER BY L. G. WILSON

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1. INTRODUCTION

Wilson [3] uses the expression (2.1) below, which approximates the Fibonacci and Lucas sequences $\{F_r\}$ and $\{L_r\}$, respectively, for r sufficiently large. The object of this paper is to make known this and another expression (3.1) by applying techniques different from those used in [3]. In particular, we need

$$\beta_i = 4\cos^2\frac{i\pi}{2n}. \tag{1.1}$$

Special attention is directed to the sequence (2.4).

2. A GENERATING EXPRESSION

Consider

$$F_r(x, y) \equiv T_r = \left(\frac{x + \sqrt{x^2+4x}}{2}\right)^{r-1} y^{-1/2}, \tag{2.1}$$

in which x and y are real numbers and $r \to \infty$. Some applications of this expression are given in Examples 1-3.

EXAMPLE 1: Let $x = 1$ and $y = 5$, then

$$F_{r+1}(1,\ 5) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^r = \frac{\alpha^r}{\sqrt{5}}, \text{ where } \alpha = \frac{1+\sqrt{5}}{2}. \tag{2.2}$$

Using Binet's formula, we see that

$$\lim_{r\to\infty} F_{r+1}(1,\ 5) = F_r.$$

EXAMPLE 2: Let $x = 1$ and $y = \dfrac{3-\sqrt{5}}{2} = 4\cos^2\dfrac{2\pi}{5}$, then $y^{-1} = \alpha^2$. Hence,

$$\begin{cases} F_{r+1}(1,\ 1) = \left(\dfrac{1+\sqrt{5}}{2}\right)^r = \alpha^r \\ \lim\limits_{r\to\infty} F_{r+1}(1,\ 1) = L_r. \end{cases} \tag{2.3}$$

EXAMPLE 3: Let x be the real root of $t^3 + t^2 - 1 = 0$ and $y = x$. It can be verified that

$$x = \sqrt[3]{\frac{25}{54} + \sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{25}{54} - \sqrt{\frac{23}{108}}} - \frac{1}{3},$$

and it is shown in [2] that the reciprocal of x is the real root of the characteristic equation for (2.5) below. For r sufficiently large, (2.1) approximates the Neumann sequence discussed in [2] and [3] and given by

(2.4)

T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	...
3	0	2	3	2	5	5	7	10	12	17	22	29	39	51	68	90	119	...

where

$$T_n = T_{n-2} + T_{n-3} \qquad (n \ge 3). \tag{2.5}$$

This is possibly the slowest growing integer sequence for which $p|T_p$ for all prime (see [2]).

3. COMPLEX SEQUENCES

Write

$$F_{mn} = \left(\frac{x + \sqrt{x^2 + 4x}}{2}\right)^n, \tag{3.1}$$

where

$$x = z_m = -\beta_m = -4\cos^2\frac{m\pi}{2n} \text{ [by (1.1)]},\ m = 1,\ 2,\ \ldots,\ n-1. \tag{3.2}$$

Then

$$\begin{cases} F_{mn} = (-1)^n 2^n \cos^{2n}\left(\dfrac{m\pi}{2n}\right)\left\{1 + i\tan\dfrac{m\pi}{2n}\right\} \\ \quad = (-1)^n 2^n \cos^n\left(\dfrac{m\pi}{2n}\right)e^{im\pi/2} \qquad \text{by Euler's Theorem} \\ \quad = (-1)^n \beta_m^{n/2}\, e^{im\pi/2} \end{cases} \tag{3.3}$$

so

$$\begin{cases} \sum_{m=1}^{n-1} F_{mn}^2 & = \sum_{m=1}^{n-1} (-1)^{2n} \beta_m^n e^{im\pi} \\ & = \pm \sum_{m=1}^{n-1} (-1)^n \beta_m^n \qquad \text{according as } n \text{ is } \begin{cases}\text{odd}\\ \text{even}\end{cases} \\ & = \pm \sum_{m=1}^{n-1} z_m^n \text{ [by (3.2)], according as } n \text{ is } \begin{cases}\text{odd}\\ \text{even.}\end{cases} \end{cases}$$

When $n = 6$, (3.3) gives

$$F_{16} = (26 + 15\sqrt{3})i,\ F_{26} = -27,\ F_{36} = -8i,\ F_{46} = 1,\ F_{56} = (26 - 15\sqrt{3})i.$$

From (3.2),

$$z_1 = -(2 + \sqrt{3}),\ z_2 = -3,\ z_3 = -2,\ z_4 = -1,\ z_5 = -(2 - \sqrt{3}).$$

Hence, by (3.4),

$$\sum_{m=1}^{5} F_{m6}^2 = -\sum_{m=1}^{5} z_m^6 = -6^6.$$

From (3.3), it is clear that the F_{mn} are, alternately, purely real and purely imaginary.

Together, (2.1) and (3.1) yield

$$T_{r+1} = F_{mn}^{r/n} y^{-1/2}. \tag{3.5}$$

Wilson [3] also gives the cases $y = -1$ and $y = -3$ with $n = 6$, $m = 1$, so

$$x = -4\cos^2\frac{\pi}{12} = -(2 + \sqrt{3}) \quad \text{[by (3.2)]}.$$

This produces what he calls "regular complex Fibonacci sequences," by which he means that terms at regular intervals are either purely real or purely imaginary (while in all other cases the terms are of the form $a + ib$, where a, b are real). The period of these "cycles" is, in both cases, 6, beginning with T_1. Details of the computation involved are omitted here in the interest of brevity, and are left to the reader's curiosity.

4. CONCLUDING COMMENTS

Return now to Examples 1-3 in Section 2.

In (3.5), take $n = 5$, $y = 5$, $m = 2$, i.e., $\beta_2 = 4\cos^2\frac{\pi}{5}$ by (1.1). Then Example 1, (2.2), results.

Next, with $n = 5$, $m = 2$ again, but with $y = \beta_4 = 4\cos^2\frac{2\pi}{5}$ by (1.1), Example 2, (2.3), results.

Furthermore, observe that, when $x = 1$ and $y = 5$, the recurrence relation for $T_r = F_r(1, 5)$ in (2.2) is given by

$$T_{r+1} = \sum_{n=0}^{r} (-1)^n \binom{r}{n} T_{2r-2n+1}\left[= \frac{\alpha^r}{\sqrt{5}} = \frac{(-1 + \alpha^2)^r}{\sqrt{5}}\right]. \tag{4.1}$$

This is related to the more general

$$T_n = \sum_{r=2}^{d} a_r T_{n-r}, \tag{4.2}$$

given in Neumann and Wilson [2], where $T_0 = d$, $T_1 = 0$, $T_2 = 2a_2$, $T_3 = 3a_3$, ...

When $d = 3$, $a_2 = 1$, $a_3 = 1$, we obtain from (4.2) the Neumann sequence (2.4), which, as we have noted, can also be generated by Wilson's function (2.1).

Finally, we observe that

$$(4.3)\qquad \begin{cases} f_r = F_r(1, 5) = (-1)^r/5 \cdot F_{r+1}(1, 5) \\ \ell_r = F_r(1, 1) + (-1)^r/F_r(1, 1). \end{cases}$$

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A GENERALIZATION OF THE DIRICHLET PRODUCT

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1. INTRODUCTION

If f is the Dirichlet product of arithmetical functions g and h, then by definition

$$f(n) = \sum_{d|n} g(d)h(n/d).$$

In this paper we define a convolution of two arithmetical functions that generalizes the Dirichlet product. With this new convolution, which we shall refer to as the the "k-prime product," it is possible to define arithmetical functions which are analogs of certain well-known functions such as Euler's function $\phi(n)$, defined implicitly by the relation

$$(1.1)\qquad \sum_{d\delta=n} \phi(d) = n.$$

Other well-known functions to be considered in this paper include $\tau(n)$ and $\sigma(n)$ given by $\tau(n) = \Sigma 1$ and $\sigma(n) = \Sigma d$, where the summations are over the positive divisors of n. The familiar Moebius function $\mu(n)$ is defined as the multiplicative function with the evaluation $\mu(p) = -1$ and $\mu(p^e) = 0$ if $e > 1$, and satisfies the relation

$$(1.2)\qquad \sum_{d\delta=n} \mu(d) = \varepsilon(n) \equiv \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mu(1) = 1$, since μ is a nonzero multiplicative function. Upon applying the Moebius inversion formula to (1.1), one obtains the simple Dirichlet product representation for ϕ,

$$(1.3)\qquad \phi(n) = \sum_{d\delta=n} \mu(d)\delta.$$

Another function which may be defined by means of the Dirichlet product is $q(n)$, the characteristic function of the set Q of square-free integers,