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ON THE ENUMERATION OF CERTAIN COMPOSITIONS AND RELATED SEQUENCES OF NUMBERS

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Abstract

The numbers

$$A(m,\ k,\ s,\ r) = [\nabla^{m+1}\mathrm{E}^k\underline{(sx + r)_m}]_{x=0},$$

where $\nabla = 1 - \mathrm{E}^{-1}$, $\mathrm{E}^j f(x) = f(x + j)$, $\underline{u_x} = u_x$ when $0 \le x \le k$ and $\underline{u_x} = 0$ otherwise, $(y)_m = y(y - 1) \ldots (y - m + 1)$, are the subject of this paper. Recurrence relations, generating functions, and certain other properties of these numbers are obtained. They have many similarities with the Eulerian numbers

$$A_{m,k} = \frac{1}{m!}[\nabla^{m+1}\mathrm{E}^k\underline{x^m}]_{x=0},$$

and give in particular (i) the number $C_{m,n,s}$ of compositions of n with exactly m parts, no one of which is greater than s, (ii) the number $Q_{s,m}(k)$ of sets $\{i_1,\ i_2,\ \ldots,\ i_m\}$ with $i_n \in \{1,\ 2,\ \ldots,\ s\}$ (repetitions allowed) and showing exactly k increases between adjacent elements, and (iii) the number $Q_{s,m}(r,\ k)$ of those sets which have $i_1 = r$. Also, they are related to the numbers

$$G(m,\ n,\ s,\ r) = \frac{1}{n!}[\Delta^n(sx + r)_m]_{x=0},\ \Delta = \mathrm{E} - 1,$$

used by Gould and Hopper [11] as coefficients in a generalization of the Hermite polynomials, and to the Euler numbers and the tangent-coefficients T_m. Moreover, $\lim_{s\to\pm\infty} s^{-m}m!A(m,\ k,\ s,\ su) = A_{m,k,u}$, where

$$A_{m,k,u} = \frac{1}{m!}[\nabla^{m+1}\mathrm{E}^k\underline{(x + u)^m}]_{x=0}$$

is the Dwyer [8, 9] cumulative numbers; in particular,

$$\lim_{s\to\pm\infty} s^{-m}m!A(m,\ k,\ s) = A_{m,k},\ A(m,\ k,\ s) \equiv A(m,\ k,\ s,\ 0).$$

Finally, some applications in statistics are briefly discussed.

1. Introduction

A partition of a positive integer n is a collection of positive integers, without regard to order, whose sum is equal to n. The corresponding ordered collections are called "compositions" of n. The integers collected to form a partition (or composition) are called its "parts" (cf. MacMahon [14, Vol. I, p. 150] and Riordan [16, p. 124]). The compositions with exactly m parts, no one of which is greater than s, have generating function

$$C_{m,s}(t) = \sum C_{m,n,s}t^n = t^m(1-t)^{-m}(1-t^s)^m,$$

and therefore the number $C_{m,n,s}$ of compositions of n with exactly m parts, no one of which is greater than s, is given by the sum

$$C_{m,n,s} = \sum_{j=1}^{k} (-1)^j \binom{m}{j}\binom{n-1-sj}{m-1}, \tag{1.1}$$

where $k = [(n-m)/s]$, the integral part of $(n-m)/s$.

Compositions of this type arose in the following Montmort-Moivre problem (cf. Jordan [12, p. 140] and [13, p. 449]): Consider m urns each with s balls bearing the numbers 1, 2, ..., s. Suppose that one ball is drawn from each urn and let

$$Z = \sum_{i=1}^{m} X_i$$

be the sum of the selected numbers. Then the probability $p(n; m, s)$ that Z is equal to n is given by

$$p(n; m, s) = s^{-m}C_{m,n,s}, \quad n = m, m+1, \ldots, sm. \tag{1.2}$$

Carlitz, Roselle, and Scoville [4] proved that the number $Q_{s,m}(k)$, of sets $\{i_1, i_2, \ldots, i_m\}$ with $i_n \in \{1, 2, \ldots, s\}$ (repetitions allowed) and showing exactly k increases between adjacent elements, is given by

$$Q_{s,m}(k) = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j}\binom{s(k-j)+m-1}{m}, \tag{1.3}$$

and the number $Q_{s,m}(r, k)$ of those sets which have $i_1 = r$ is given by

$$Q_{s,m}(r, k) = \sum_{j=0}^{k-1} (-1)^j \binom{m}{j}\binom{s(k-j-1)+r+m-2}{m-1}. \tag{1.4}$$

The next problem is from applied statistics: Dwyer [8,9] studied the problem of computing the ordinary moments of a frequency distribution with the use of the cumulative totals and certain sequences of numbers. These numbers are the coefficients $A_{m,k,r}$ of the expansion of $(x+r)^m$ into a series of factorials $(x+k)_m$, $k = 0, 1, 2, \ldots, m$; that is,

$$(x+r)^m = \sum_{k=0}^{m} A_{m,k,r}(x+m-k)_m/m!$$

Using the notation $\underline{u_x} = u_x$ with $0 \leq x \leq k$ and $\underline{u_x} = 0$ otherwise, he proved that

(1.5) $$A_{m,k,r} = [\nabla^{m+1}E^k\underline{(x + r)^m}]_{x=0} = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}(k - j + r)^m.$$

These numbers for $r = 0$ reduce to the Eulerian numbers

(1.6) $$A_{m,k} = [\nabla^{m+1}E^k\underline{x}^m]_{x=0} = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}(k - j)^m.$$

In the present paper, starting from the problem of computing the factorial moments of a frequency distribution with the use of cumulative totals, we introduce the numbers

(1.7) $$A(m, k, s, r) = \frac{1}{m!}[\nabla^{m+1}E^k\underline{(sx + r)_m}]_{x=0}$$

so that

(1.8) $$(sx + r)_m = \sum_{k=0}^{m}A(m, k, s, r)(x + m - k)_m.$$

These numbers have many similarities with the Eulerian numbers (cf. Carlitz [1]). They are related to the numbers $C_{m,n,s}$ of the title of this paper by

$$C_{m,n,s} = A(m - 1, k - 1, s, r + m - 1) = (-1)^{m-1}A(m - 1, k - 1, -s, -r - 1),$$

$$k = [(n - m)/s],$$

$$r = (n - m) - s[(n - m)/s],$$

and to the numbers $Q_{s,m}(r, k)$ by

$$Q_{s,m}(r, k) = A(m - 1, k - 1, s, r + m - 2) = (-1)^{m-1}A(m - 1, k - 1, -s, -r),$$

and their properties are discussed in Sections 2 and 3 below. Since

$$Q_{s,m}(k) = Q_{s,m+1}(s, k),$$

it follows that

$$Q_{s,m}(k) = A(m, k - 1, s, s + m - 1) = (-1)^m A(m, k, -s),$$

where $A(m, k, s) \equiv A(m, k, s, 0)$. Section 4 is devoted to the discussion of certain statistical applications of the numbers $A(m, k, s, r)$.

2. The Composition Numbers $A(m, k, s, r)$

Let $(x)_{m,b} = x(x - b)\ldots(x - mb + b)$ denote the generalized falling factorial of degree m with increment b; the usual falling factorial of degree m will be denoted by $(x)_m = (x)_{m,1}$. The problem of expressing the generalized factorial $(x)_{m,b}$ in terms of the generalized factorials $(x + ka)_{m,a}$, $k = 0, 1, 2, \ldots, m$ of the same degree arises in statistics in connection with the problem of expressing the generalized factorial moments in terms of the cumulations (see Dwyer [8, 9] and Section 4 below). More generally, let

(2.1) $$(x + rb)_{m,b} = \sum_{k=0}^{m}C_{m,k,r}(a, b)(x + (m - k)a)_{m,a}.$$

Following Dwyer, define $\underline{u_x} = u_x$ when $0 \le x \le k$ and $\underline{u_x} = 0$ otherwise. Moreover, let E_a denote the displacement operator defined by $\mathrm{E}_a f(x) = f(x + a)$ and $\nabla_a = 1 - \mathrm{E}_a^{-1}$, the receding difference operator; when $a = 1$, we write $\mathrm{E}_1 \equiv \mathrm{E}$ and $\nabla_1 \equiv \nabla$. Then, from (2.1), we have

$$(2.2) \qquad \underline{(x + rb)}_{m,b} = \sum_{k=0}^{m} C_{m,k,r}(a, b)\underline{(x + (m - k)a)}_{m,a}.$$

Since

$$[\nabla^{m+1}\mathrm{E}_a^n(x + (m - k)a)_{m,a}]_{x=0} = \begin{cases} a^m m!, & k = n \\ 0, & 0 \le k < n \quad \text{or} \quad n < k \le m, \end{cases}$$

we get, from (2.2)

$$C_{m,k,r}(a, b) = \frac{a^{-m}}{m!}[\nabla_a^{m+1}\mathrm{E}_a^k\underline{(x + rb)}_{m,b}]_{x=0}.$$

These coefficients may be expressed in terms of the operators ∇ and E and the usual falling factorials by using the relations

$$\nabla_a^{m+1}\mathrm{E}_a^k f(x) = \nabla^{m+1}\mathrm{E}^k f(ax), \quad (ax + rb)_{m,b} = b^m(sx + r)_m, \quad s = a/b.$$

We find

$$C_{m,k,r}(a, b) = s^{-m}A(m, k, s, r), \quad s = a/b,$$

where

$$(2.3) \qquad A(m, k, s, r) = \frac{1}{m!}[\nabla^{m+1}\mathrm{E}^k\underline{(sx + r)}_m]_{x=0}, \quad k = 0, 1, \ldots, m, \quad m = 0, 1, 2, \ldots .$$

Hence

$$(2.4) \qquad (x + rb)_{m,b} = \sum_{k=0}^{m} s^{-m}A(m, k, s, r)(x + (m - k)a)_{m,a}, \quad s = a/b$$

or

$$(2.5) \qquad (ax + r)_m = \sum_{k=0}^{m} A(m, k, s, r)(bx + m - k)_m.$$

Using the symbolic formula

$$\nabla^{m+1} = \sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\mathrm{E}^{-j},$$

we get, for the numbers (2.3), the explicit expression

$$(2.6) \qquad A(m, k, s, r) = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)+r}{m}.$$

It is easily seen that

$$(2.7) \qquad A(m, k, -s, -r) = (-1)^m A(m, k, s, r + m - 1),$$

and, also, that the numbers $A(m, k, s, r)$ are integers when s and r are integers. Moreover, $A(m, k, s, r) = 0$ when $k > m$.

Remarks 2.1: As we have noted in the introduction, the number $C_{m,n,s}$ of compositions of n with exactly m parts, none of which is greater than s is given by

$$(2.8)\qquad C_{m,n,s} = \sum_{j=0}^{k}(-1)^j\binom{m}{j}\binom{n-sj-1}{m-1},\ k = [(n-m)/s].$$

Comparing (2.8) with (2.6) and using (2.7), we get the relation

$$(2.9)\qquad \begin{aligned} C_{m,n,s} &= A(m-1,\ k,\ s,\ r+m-1) \\ &= (-1)^{m-1}A(m-1,\ k,\ -s,\ -r-1),\ r = (n-m) - s[(n-m)/s], \end{aligned}$$

which justifies the title of this section.

Since the number $Q_{s,m}(r, k)$, of sets $\{i_1, i_2, \ldots, i_m\}$ with $i_n \in \{1, 2, \ldots, s\}$ (repetitions allowed) and showing exactly k increases between adjacent elements which have $i_1 = r$, is given by (see [4])

$$(2.10)\qquad Q_{s,m}(r,k) = \sum_{j=0}^{k-1}(-1)^j\binom{m}{j}\binom{s(k-1-j)+r+m-2}{m-1},$$

we get, by virtue of (2.6), the relation

$$(2.11)\qquad \begin{aligned} Q_{s,m}(r,k) &= A(m-1,\ k-1,\ s,\ r+m-2) \\ &= (-1)^{m-1}A(m-1,\ k-1,\ -s,\ -r). \end{aligned}$$

These numbers give in particular the numbers

$$(2.12)\qquad Q_{s,m}(k) = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)+m-1}{m}$$

of the above sets without the restriction $i_1 = r$. We have

$$Q_{s,m}(k) = Q_{s,m+1}(s,k),$$

and hence

$$(2.13)\qquad Q_{s,m}(k) = A(m,\ k-1,\ s,\ s+m-1) = (-1)^m A(m,\ k,\ -s),$$

where

$$(2.14)\qquad A(m,k,s) = \frac{1}{m!}[\nabla^{m+1}E^k(sx)_m]_{x=0} = \sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)}{m}.$$

Since

$$\sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\binom{s(k-j)+r}{m} = \left[\Delta^{m+1}\binom{-sx+r}{m}\right]_{x=-k} = 0,$$

it follows that

$$\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}\binom{s(k-j)+r}{m} = \sum_{j=k+1}^{m+1}(-1)^{j+1}\binom{m+1}{j}\binom{s(k-j)+r}{m}$$

and (2.6) may be rewritten as follows:

$$\begin{aligned}A(m, k, s, r) &= \sum_{j=k+1}^{m+1} (-1)^{j+1}\binom{m+1}{j}\binom{s(k-j)+r}{m} \\ &= \sum_{i=0}^{m-k} (-1)^{m+i}\binom{m+1}{i}\binom{-s(m-k+1-i)+r}{m} \\ &= \sum_{i=0}^{m-k+1} (-1)^{i}\binom{m+1}{i}\binom{s(m-k+1-i)+m-r-1}{m} \\ &\qquad + (-1)^k\binom{m+1}{m-k+1}\binom{r}{m}.\end{aligned}$$

Hence

$$\begin{aligned}A(m, k, s, r) &= A(m, m-k+1, s, m-r-1) + (-1)^k\binom{m+1}{m-k+1}\binom{r}{m} \\ &= (-1)^m A(m, m-k+1, -s, r) + (-1)^k\binom{m+1}{m-k+1}\binom{r}{m}.\end{aligned} \tag{2.15}$$

In particular

$$A(m, k, s) = (-1)^m A(m, m-k+1, -s), \tag{2.16}$$

which should be compared with the symmetric property of the Eulerian numbers $A_{m,k} = A_{m,m-k+1}$.

Using the relation

$$\binom{m+2}{j}(s(k-j)+r-m) = (sk-m+r)\binom{m+1}{j} - (s(m-k+2)+m-r)\binom{m+1}{j-1},$$

we get, from (2.6), the recurrence relation

$$\begin{aligned}&(m+1)A(m+1, k, s, r) \\ &\quad = (sk-m+r)A(m, k, s, r) + (s(m-k+1)+m-r)A(m, k-1, s, r)\end{aligned} \tag{2.17}$$

with initial conditions

$$A(0, 0, s, r) = 1,\ A(m, 0, s, r) = \binom{r}{m},\ m > 0.$$

From (2.5), we have

$$(s(k-j)+r)_m = \sum_{i=0}^{m} A(m, m-i, k-j, r)(s+i)_m.$$

Hence (2.6) may be rewritten as

$$\begin{aligned}A(m, k, s, r) &= \sum_{j=0}^{k} (-1)^j\binom{m+1}{j}\binom{s(k-j)+r}{m} \\ &= \sum_{j=0}^{k} (-1)^j\binom{m+1}{j}\sum_{i=0}^{m}\binom{s+i}{m}A(m, m-i, k-j, r) \\ &= \sum_{i=0}^{m}\binom{s+i}{m}\sum_{j=0}^{k}(-1)^j\binom{m+1}{j}A(m, m-i, k-j, r),\end{aligned}$$

and putting

$$B(m, n, k, r) = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} A(m, n, k-j, r) \tag{2.18}$$

we get

$$A(m, k, s, r) = \sum_{i=0}^{m} \binom{s+i}{m} B(m, m-i, k, r), \tag{2.19}$$

and

$$B(m, n, k, r) = \sum_{j=0}^{k} \sum_{i=0}^{n} (-1)^{i+j} \binom{m+1}{i} \binom{m+1}{j} \binom{(n-i)(k-j)+r}{m}. \tag{2.20}$$

It is clear from (2.20) that

$$B(m, n, k, r) = B(m, k, n, r). \tag{2.21}$$

Since

$$\begin{aligned}
&\binom{m+2}{i}\binom{m+2}{j}((n-i)(k-j)+r-m) \\
&= (nk-m+r)\binom{m+1}{i}\binom{m+1}{j} - (n(m-k+2)+m-r)\binom{m+1}{i}\binom{m+1}{j-1} \\
&\quad - (k(m-n+2)+m-r)\binom{m+1}{i-1}\binom{m+1}{j} \\
&\quad + ((m-n+2)(m-k+2)-m+r)\binom{m+1}{i-1}\binom{m+1}{j-1},
\end{aligned}$$

it follows, from (2.20), that

$$\begin{aligned}
(2.22)\quad &(m+1)B(m+1, n, k, r) \\
&= (nk-m+r)B(m, n, k, r) + (n(m-k+2)+m-r)B(m, n, k-1, r) \\
&\quad + (k(m-n+2)+m-r)B(m, n-1, k, r) \\
&\quad + ((m-n+2)(m-k+2)-m+r)B(m, n-1, k-1, r).
\end{aligned}$$

with

$$B(0, 0, 0, r) = 1, \; B(0, 0, k, r) = B(0, k, 0, r) = 0.$$

Remark 2.2: Comparing (2.20) with the formula

$$R_m(n, k) = \sum_{j=0}^{k} \sum_{i=0}^{n} (-1)^{i+j} \binom{m+1}{i} \binom{m+1}{j} \binom{(n-i)(k-j)+m-1}{m}$$

giving the number of permutations on m letters which have n jumps and require k readings (cf. [4]), we find

$$\begin{aligned}
(2.23)\quad R_m(n, k) &= B(m, n, k, m-1) = B(m, m-n+1, k) \\
&= B(m, n, m-k+1),
\end{aligned}$$

where

$$B(m, n, k) \equiv B(m, n, k, 0) = \sum_{j=0}^{k} \sum_{i=0}^{n} (-1)^{i+j} \binom{m+1}{i} \binom{m+1}{j} \binom{(n-i)(k-j)}{m}. \tag{2.24}$$

Using the relation

$$\sum_{j=0}^{k}\sum_{i=0}^{n}(-1)^{i+j}\binom{m+1}{i}\binom{m+1}{j}\binom{(n-i)(k-j)}{m}$$
$$=\sum_{j=k+1}^{m+1}\sum_{i=n+1}^{m+1}(-1)^{i+j}\binom{m+1}{i}\binom{m+1}{j}\binom{(n-i)(k-j)}{m},$$

it can be easily shown that

$$B(m,\ n,\ k) = B(m,\ m-n+1,\ m-k+1). \tag{2.25}$$

Expanding the generalized factorial $(x+rb)_{m,b}$ in terms of the generalized factorials $(x+ka)_{m,a}$, $k = 0, 1, 2, \ldots, m$ and then these factorials in terms of the factorials $(x+jb)_{m,b}$, $j = 0, 1, 2, \ldots, m$, by using (2.4), we get

$$\begin{aligned}(x+rb)_{m,b} &= \sum_{k=0}^{m} a^{-m}b^{m}A(m,\ m-k,\ a/b,\ r)(x+ka)_{m,a}\\ &= \sum_{k=0}^{m}\sum_{j=0}^{m} A(m,\ m-k,\ a/b,\ r)A(m,\ m-j,\ b/a,\ k)(x+jb)_{m,b}\\ &= \sum_{j=0}^{m}\sum_{k=0}^{m} A(m,\ m-k,\ a/b,\ r)A(m,\ m-j,\ b/a,\ r)(x+jb)_{m,b},\end{aligned}$$

which implies

$$\sum_{k=0}^{m} A(m,\ m-k,\ a/b,\ r)A(m,\ m-j,\ b/a,\ r) = \delta_{rj}, \tag{2.26}$$

with δ_{rj} the Kronecker delta: $\delta_{rr} = 1$, $\delta_{rj} = 0$, $j \neq r$. Hence, we have the pair of inverse relations

$$\alpha_r = \sum A(m,\ m-k,\ a/b,\ r)\beta_k,\quad \beta_k = \sum A(m,\ m-k,\ b/a,\ r)\alpha_r. \tag{2.27}$$

3. Generating Functions and Connection with Other Sequences of Numbers

Consider first the generating function

$$A_{m,s,r}(t) = \sum A(m,\ k,\ s,\ r)t^k, \tag{3.1}$$

where the summation is over all possible values of k which are 0 to m and can be left indefinite because $A(m, k, s, r)$ is zero elsewhere. Then, from (2.6), it follows that

$$A_{m,s,r}(t) = (1-t)^{m+1}\sum_{k=0}^{\infty}\binom{sk+r}{m}t^k. \tag{3.2}$$

In a generalization of the Hermite polynomials, Gould and Hopper [11] used as coefficients the numbers

$$G(m,\ n,\ s,\ r) = \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}(sj+r)_m, \tag{3.3}$$

which may be equivalently defined by

$$G(m, n, s, r) = \frac{1}{n!}[\Delta^n (sx + r)_m]_{x=0}.$$

Using the symbolic formula

$$\mathrm{E}^k = \sum_{n=0}^{\infty} \binom{k}{n} \Delta^n$$

and since $[\mathrm{E}^k (sx + r)_m]_{x=0} = (sk + r)_m$, we get

$$(sk + r)_m = \sum_{n=0}^{m} G(m, n, s, r)(k)_n. \tag{3.4}$$

The generating function (3.2) may then be rewritten as

$$A_{m,s,r}(t) = \sum_{n=0}^{m} \frac{n!}{m!} G(m, n, s, r) t^m (1 - t)^{m-n},$$

so that

$$A(m, k, s, r) = \sum_{n=0}^{m} (-1)^{k-n} \frac{n!}{m!} \binom{m - n}{m - k} G(m, n, s, r). \tag{3.5}$$

Since for $r = 0$ the numbers $G(m, n, s, r)$ reduce to the numbers

$$C(m, n, s) = \frac{1}{n!}[\Delta^n (sx)_m]_{x=0}$$

studied by the author [5, 6, 7] and also by Carlitz [2] as degenerate Stirling numbers, we have, in particular,

$$A(m, k, s) = \sum_{n=0}^{m} (-1)^{k-n} \frac{n!}{m!} \binom{m - n}{m - k} C(m, n, s). \tag{3.6}$$

The generating functions

$$A_{s,r}(t, x) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A(m, k, s, r) t^k x^m \tag{3.7}$$

and

$$A_s(t, x, y) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} A(m, k, s, r) t^k y^r x^m, \tag{3.8}$$

using (3.2), may be obtained as

$$A_{s,r}(t, x) = \frac{(1 - t)[1 + (1 - t)x]^r}{1 - t[1 + (1 - t)x]^s}, \tag{3.9}$$

$$A_s(t, x, y) = \frac{(1 - t)}{\{1 - t[1 + (1 - t)x]^s\}\{1 - y[1 + (1 - t)x]\}}. \tag{3.10}$$

Since $\lim_{t \to 1} A_{s,r}(t, x) = (1 - sx)^{-1}$, we get

$$A_{m,s,r}(1) = \sum_{k=0}^{m} A(m, k, s, r) = s^m. \tag{3.11}$$

Using (2.19) and (2.21), (3.11) may be rewritten in the form

$$\begin{aligned} s^m &= \sum_{k=0}^{m} \sum_{i=1}^{m+1} \binom{s+i-1}{m} B(m,\ m-i+1,\ k,\ r) \\ &= \sum_{k=0}^{m} \sum_{i=1}^{m+1} \binom{s+i-1}{m} B(m,\ i,\ m-k+1,\ r) \\ &= \sum_{i=1}^{m+1} \binom{s+i-1}{m} \sum_{k=0}^{m} B(m,\ i,\ m-k+1,\ r). \end{aligned}$$

It is known that the Eulerian numbers $A_{m,i}$ satisfy the relation (see [19] or [1])

$$\sum_{i=1}^{m} \binom{s+i-1}{m} A_{m,i} = s^m.$$

Therefore

$$\sum_{k=1}^{m+1} B(m,\ i,\ k,\ r) = A_{m,i}. \tag{3.12}$$

The generating function

$$B_{m,n,r}(t) = \sum_{k=0}^{m} B(m,\ n,\ k,\ r)t^k \tag{3.13}$$

is connected with $A_{m,s,r}(t)$ by the relations

$$A_{m,s,r}(t) = \sum_{i=0}^{m} \binom{s+i}{m} B_{m,m-i,r}(t) \tag{3.14}$$

and

$$B_{m,n,r}(t) = \sum_{j=0}^{n} (-1)^j \binom{m+1}{j} A_{m,n-j,r}(t). \tag{3.15}$$

Returning to (2.6), let us put $r = su$. Then

$$\lim_{s \to \pm\infty} s^{-m} m! A(m,\ k,\ s,\ r) = A_{m,k,u}, \tag{3.16}$$

where

$$A_{m,k,u} = [\nabla^{m+1} \mathrm{E}^k \underline{(x+u)}^m]_{x=0} = \sum_{j=0}^{k} (-1)^j \binom{m+1}{j} (k+u-j)^m$$

are the numbers used by Dwyer [8] for computing the ordinary moments of a frequency distribution. In particular,

$$\lim_{s \to \pm\infty} s^{-m} m! A(m,\ k,\ s) = A_{m,k}. \tag{3.17}$$

Consider the function

$$\begin{aligned} H_m(t;\ s,\ r) &= (1-t)^{-m} A_{m,s,r}(t) \\ &= \sum_{n=0}^{m} \frac{n!}{m!} G(m,\ n,\ s,\ r)t^n (1-t)^{-n}. \end{aligned} \tag{3.18}$$

Then, using (3.3), we get

$$H(x;\ t,\ s,\ r) = \sum_{m=0}^{\infty} H_m(t;\ s,\ r)x^m = (1-t)(1+x)^r[1-t(1+x)^s]^{-1}. \tag{3.19}$$

Since $\lim_{s \to \pm\infty} H(x/s; -1, s, su) = E(x; u)$, where

$$E(x; u) = \sum_{m=0}^{\infty} E_m(u)x^m = 2e^{xu}/(1 + e^x) \tag{3.20}$$

is the generating function of the Euler polynomials ([12, p. 309]), it follows that for the polynomials

$$\zeta_m(u; s) \equiv H_m(-1; s, su) = 2^{-m} \sum_{k=0}^{m} (-1)^k A(m, k, s, su)$$

we have

$$\lim_{s \to \pm\infty} s^{-m}\zeta_m(u, s) = E_m(u),$$

which, on using (3.16), gives

$$\sum_{k=0}^{m} (-1)^k A_{m,k,u} = 2^m m! E_m(u) \tag{3.21}$$

and, in particular,

$$\sum_{k=0}^{m} (-1)^k A_{m,k,1/2} = E_m, \tag{3.22}$$

where $E_m = 2^m m! E_m(1/2)$ is the Euler number ([12, p. 300]).
Putting $u = 0$ in (3.21), we get

$$\sum_{k=0}^{m} (-1)^k A_{m,k} = T_m, \tag{3.23}$$

where $T_m = 2^m m! E_m(0)$ is the tangent-coefficient ([12, p. 298]).

Remark 3.1: The degenerate Eulerian numbers $A_{m,k}(\lambda)$ introduced by Carlitz [2, 3] by their generating function

$$1 + \sum_{m=1}^{\infty} \frac{x^m}{m!} \sum_{k=1}^{m} A_{m,k}(\lambda)t^k = \frac{1 - t}{1 - t[1 + \lambda x(1 - t)]^{1/\lambda}} \tag{3.24}$$

are related to the numbers

$$A(m, k, s) \equiv A(m, k, s, 0).$$

Indeed, comparing (3.24) with (3.9), we get

$$A(m, k, s) = \frac{s^m}{m!} A_{m,k}(s^{-1}).$$

4. Applications in Statistics

The numbers $A(m, k, s, r)$ like the Eulerian numbers $A_{m,k}$ seem to have many applications in combinatorics and statistics. Special cases of these numbers have already occurred in certain combinatorial problems, as was noted in the introduction. In this section, we briefly discuss three applications in statistics. The first is in the computation of the factorial moments of a frequency distribution with the use of cumulative totals. This method was suggested by Dwyer [8, 9] for the computation of the ordinary moments, as an alternative to the usual elementary method and, therefore, for details, the reader is referred to this work. We only note that the main advantage of this method is that the many multiplications involved in the usual process are replaced by continued addition. Let f_x denote the frequency distribution and

$$C^{m+1}f_x = C(C^m f_x),\ m = 1, 2, 3, \ldots,\ Cf_x = \sum_{j=x}^{r+k} f_j,$$

the successive frequency cumulations. Then, from the successive cumulation theorem of Dwyer, we get, for the factorial moments,

$$\sum_{x=0}^{k} (sx + r)_m f_{sx+r} = \sum_{n=0}^{m} m!A(m, n, s, r)C^{m+1}f_{sn+r}. \tag{4.1}$$

When $r = 0$, i.e., when the factorial moments are measured about the smallest variate, (4.1) reduces to

$$\sum_{x=0}^{k} (sx)_m f_{sx} = \sum_{n=0}^{m} m!A(m, n, s)C^{m+1}f_{sn}, \tag{4.2}$$

which for $s = 1$, i.e., when the distance between successive variates (class marks) is unity, gives ([8, §9])

$$\sum_{x=0}^{k} (x)_m f_x = \sum_{n=0}^{m} m!A(m, n, 1)C^{m+1}f_n = m!C_{m+1}^{m+1}, \tag{4.3}$$

since $A(m, m, 1) = 1$, $A(m, n, 1) = 0$ if $n \neq m$.

The second statistical application of the numbers $A(m, k, s, r)$ is in the following problem: Let $X_1, X_2, \ldots, X_m$ be a random sample (that is, m independent and identically distributed random variables) from a population with a discrete uniform distribution

$$p(n; s) = P(X = n) = s^{-1},\ n = 0, 1, 2, \ldots, s - 1.$$

Then the probability function of the sum $Z_m = \sum_{i=1}^{m} X_i$ may be obtained as

$$\begin{aligned} p(n; m, s) &= s^{-m} \sum_{j=0}^{[n/s]} (-1)^j \binom{m}{j}\binom{n + m - 1 - sj}{m - 1} \\ &= s^{-m}A(m - 1, [n/s], s, r + m - 1), \\ &\qquad n = sk + r,\ 0 \le r < s. \end{aligned} \tag{4.4}$$

Note that the distribution function

$$F_{m,s}(w) = \sum_{n=0}^{[w]} p(n/s; m, s)$$

of the sum

$$W_m = \sum_{i=1}^{m} Y_i,\ Y_i = s^{-1}X_i,\ i = 1, 2, \ldots, m$$

approaches, for $s \to \infty$, the distribution function

$$F_m(u) = \frac{1}{m!} \sum_{j=0}^{[u]} (-1)^j \binom{m}{j} (u - j)^m$$

of the sum

$$U_m = \sum_{i=1}^{m} V_i$$

of m independent continuous uniform random variables on [0, 1) (Feller [10] and Tanny [18]).

Since

$$E(Z_m) = mE(X) = m(s - 1)/2,$$
$$\mathrm{Var}(Z_m) = m\mathrm{Var}(X) = m(s^2 - 1)/12,$$

it follows from the central limit theorem (see, e.g., Feller [10]) that the sequence

$$\frac{Z_m - m(s - 1)/2}{\sqrt{m(s^2 - 1)/12}}$$

converges in distribution to the standard normal. Hence

$$\lim_{m\to\infty} \sum_{k=0}^{[z_m]} s^{-m}A(m - 1, k, s, r) = \Phi(z), \tag{4.5}$$
$$z_m = z\sqrt{m(s^2 - 1)/12} + m(s - 1)/2.$$

and

$$\lim_{m\to\infty} \sqrt{m(s^2 - 1)/12}\, s^{-m}A(m - 1, [z_m], s, r + m - 1) = \varphi(z), \tag{4.6}$$

where $\varphi(z)$ and $\Phi(z)$ are the density and the cumulative distribution functions of the standard normal.

Finally, consider a random variable X with the logarithmic series distribution

$$p(k;\, \theta) = P(X = k) = \alpha\theta^k/k,\ k = 1, 2, \ldots, \alpha^{-1} = -\log(1 - \theta),\ 0 < \theta < 1.$$

Patil and Wani [15] proved the following property of the moments $\mu_m(\theta) = E(X^m)$:

$$\mu_m(\theta) = \alpha(1 - \theta)^{-m} \sum_{k=0}^{\infty} c(m - 2, k)\theta^{k+1},$$

where the coefficients satisfy the recurrence relation

$$c(m, k) = (k + 1)c(m - 1, k) + (m - k + 1)c(m - 1, k - 1),$$
$$c(0, 0) = 1, \quad c(m, k) = 0,\ k > m.$$

It is not difficult to see that

$$c(m, k) = A_{m+1,\, k+1}$$

with the latter a Eulerian number. Hence

$$\mu_m(\theta) = \alpha(1 - \theta)^{-m} \sum_{k=0}^{\infty} A_{m-1,\, k}\theta^k = \alpha(1 - \theta)^{-m}A_{m-1}(\theta).$$

A similar result can be obtained for the generalized factorial moments

$$\mu_{(m;b)}(\theta) = E[(X)_{m,b}]$$

in terms of the numbers $A(m, k, s, r)$. Indeed, we have

$$\mu_{(m;b)}(\theta) = \alpha \sum_{k=1}^{\infty} (k)_{m,b}\theta^k/k = \frac{\alpha}{m}\sum_{k=1}^{\infty}(k - b)_{m,b}\theta^k$$

$$= \alpha s^{-m+1}\sum_{k=1}^{\infty}(sk - 1)_{m-1}\theta^k,\ s = b^{-1},$$

and since, by (2.17) and (2.18),

$$\sum_{k=1}^{\infty}(sk + r)_{m-1}t^k = (m - 1)!(1 - t)^{-m}A_{m-1,s,r}(t),$$

it follows that

$$\mu_{(m;b)}(\theta) = \alpha s^{-m+1}(1 - \theta)^{-m}(m - 1)!A_{m-1,s,-1}(\theta), \tag{4.7}$$

which, in particular, gives

$$\mu_{(m;1)}(\theta) = \alpha(1 - \theta)^{-m}(m - 1)!\sum_{k=1}^{m-1}A(m - 1, k, 1, -1)\theta^k$$

$$= \alpha(m - 1)!\theta^m(1 - \theta)^{-m}.$$

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A GENERALIZATION OF THE GOLDEN SECTION

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Introduction

It may surprise some people to find that the name "golden section," or, more precisely, *goldener Schnitt*, for the division of a line AB at a point C such that $AB \cdot CB = AC^2$, seems to appear in print for the first time in 1835 in the book *Die reine Elementar-Mathematik* by Martin Ohm, the younger brother of the physicist Georg Simon Ohm. By 1849, it had reached the title of a book: *Der allgemeine goldene Schnitt und sein Zusammenhang mit der harminischen Theilung* by A. Wiegang. The first use in English appears to have been in the ninth edition of the *Encyclopaedia Britannica* (1875), in an article on Aesthetics by James Sully, in which he refers to the "interesting experimental enquiry . . . instituted by Fechner into the alleged superiority of 'the golden section' as a visible proportion. Zeising, the author of this theory, asserts that the most pleasing division of a line, say in a cross, is the golden section . . . ." The first English use in a purely mathematical context appears to be in G. Chrystal's *Introduction to Algebra* (1898).

The question of when the name first appeared, in any language, was raised by G. Sarton [11] in 1951, who specifically asked if any medieval references are known. The *Oxford English Dictionary* extends Sarton's list of names and references and, by implication, answers this question in the negative. (The 1933 edition of the *OED* is a reissue of the *New English Dictionary*, which appeared in parts between 1897 and 1928, together with a Supplement. The main dictionary entry "Golden," in a volume which appeared in 1900, makes no reference to the golden section, though it does cite mathematical references that will be noted later; the entry "Section" (1910) contains a reference to "medial section" (Leslie, *Elementary Geometry and Plane Trigonometry*, fourth edition, 1820) and to Chrystal's use of "golden section" noted above. The 1933