PRIMITIVE PYTHAGOREAN TRIPLES

(Submitted August 1980)

LEON BERNSTEIN
Illinois Institute of Technology, Chicago, IL 60616

Dedicated to Carl Menger, on the occasion of his 80th birthday.

0. Introduction

This paper investigates some problems concerning PRIMITIVE PYTHAGOREAN TRIPLES (PPT) and succeeds in solving, completely or partially, some of these problems while leaving open others. Dickson [2], in his three-volume history of number theory has given a twenty-five-page account of what was achieved in the field of Pythagorean triangles during more than two millenia and up to Euler and modern times. Therefore, it is surprising that still more questions can be asked which, in their intriguing simplicity, do not lag behind anything the human mind has been occupied with since the times of Hamurabi. The author thinks that, in spite of the accelerated speed with which the modern mathematical creativeness is advancing in the era of Godel and Matajasevich, some of his unanswered questions will remain enigmatic for many decades to come.

1. Definition

There are a variety of definitions on the subject of PPTs. The author thinks that he was able to come up with some of his results thanks to a simplification of on such definition, which is as follows:

Definition 1

A triple (x, y, z) of natural numbers if a PPT iff there exists a pair (u, v) of natural numbers such that

$$x = u^2 - v^2,\ y = 2uv,\ z = u^2 + v^2, \qquad (u, v) = 1,\ u + v \equiv 1 \pmod 2. \tag{1.1}$$

The pair of numbers (u, v) as introduced in Definition 1 is called a *generator* of the PPT (x, y, z). We shall use the chain of inequalities

$$2u > u + v > u, \tag{1.2}$$

which follows from Definition 1.

All small italic letters appearing in this paper denote natural numbers, 1, 2, 3, ..., if not stated otherwise.

By virtue of Definition 1, a countability of all PPTs has been established, namely,

$$(u, v) = (2, 1) \Rightarrow (x, y, z) = (3, 4, 5);$$

$$(u, v) = (3, 2) \Rightarrow (x, y, z) = (5, 12, 13);$$

$$(u, v) = (4, 1) \Rightarrow (x, y, z) = (15, 8, 17);$$

$$(u, v) = (4, 3) \Rightarrow (x, y, z) = (7, 24, 25);$$

etc.

If we drop the condition $(u, v) = 1$ in (1.1), then the resulting triple (x, y, z) is a Nonprimitive Pythagorean Triple. They are of no interest to us.

2. Pythagorean Frequency Indicator

We introduce the interesting

Definition 2

The number of times the integer n appears in some PPT, excluding order, is called the PYTHAGOREAN FREQUENCY INDICATOR (PFI) of n. The PFI of n is denoted by $f(n)$. We write $f(n) = 2^{-\infty}$, if n does not appear in any PPT. As we shall see later,

$$f(1) = 2^{-\infty},\ f(2) = 2^{-\infty},\ f(3) = 1,$$

$$f(4) = 1,\ f(5) = 2,\ \ldots,\ f(84) = 4, \text{ etc.}$$

The following result is due to Landau [4]:

Theorem 1

The number of positive solutions $L(n)$ of $x^2 + y^2 = n$ (excluding order), with $(x, y) = 1$ and

(2.1) $$x^2 + y^2 = n = \prod_{i=1}^{k} p_i^{s_i},\ p_i \text{ an odd prime},$$

is given by

(2.2) $$L(n) = 2^{k-1} \text{ if each } p_i \equiv 1 \pmod 4$$

and

(2.3) $$L(n) = 0, \text{ if at least one } p_i \equiv 3 \pmod 4.$$

Landau's theorem also elaborates on such numbers n which are not of the form (2.1) with (2.2) or (2.3), but that is not relevant for us. To state the main theorem of this chapter, it is useful to introduce the following.

Let n be as in (2.1). If all primes are as in (2.2), we let $n = 0(k, 1)$; otherwise, we let $n = 0(k, 3)$.

Theorem 2

The PFI of any number n equals

$$\begin{aligned} f(1) &= f(2) = f(2\ 0(k,1)) = f(2\ 0(k,3)) = 2^{-\infty} \\ f(0(k+1,3)) &= f(0(k,1)) = f(2^{s+1}\ 0(k,1)) \\ &= f(2^{s+1}\ 0(k,3)) = 2^k,\ s \geq 1. \end{aligned} \tag{2.4}$$

Proof: We have, by Definition 1, $x \equiv z \equiv 1 \pmod 2$, $y \equiv 0 \pmod 4$, and $x, z \neq 1$. This proves the first line of (2.4). When $n = 0(k + 1, 3)$, then only $n = x$ is possible by Theorem 1. Let $n = fg$ with $(f, g) = 1$ and $f > g$. Since $(u, v) = 1$ and $n = (u - v)(u + v)$, we have

$$u = \frac{1}{2}(f + g) \quad \text{and} \quad v = \frac{1}{2}(f - g).$$

But,

$$\binom{k+1}{0} + \binom{k+1}{1} + \cdots + \binom{k+1}{k+1} = 2^{k+1}$$

is the total number of pairs $\{f, g\}$ with $(f, g) = 1$. Hence, we have only 2^k pairs with $f > g$. Now let $n = 0(k, 1)$. When $n = z$ there are, by Theorem 2, 2^{k-1} pairs (u, v) such that $u^2 + v^2 = n$; when $n = x$ there are 2^{k-1} pairs, by the same argument given for $n = 0(k + 1, 3)$. Hence, $f(\ 0(k, 1)\ ) = 2^k$. Let $n = y = 2^{s+1}0(k, 0)$ or $n = 2^{s+1}0(k, 3)$. Let $n = 2^{s+1}fg$, where $(f, g) = 1$. Since there are only 2^{k-1} pairs (f, g), excluding order, with $(f, g) = 1$, we can choose $u = 2^s f$ and $v = g$ or $u = f$ and $v = 2^s g$. Hence, there are 2^k possibilities. This proves the second line of Theorem 2, and proves the theorem completely.

Theorem 2 also holds for $n = 2^{s+1}$ with the symbolism $n = 2^{s+1}p$, since $f(2^{s+1}) = 2^0 = 1$. The following examples illustrate the use of Theorem 2:

$f(2^{s_1+1}) = f(p^{s_2}) = 2^0 = 1$, p any odd prime, $p \equiv 3 \pmod 4$.

$f(2^{s_1+1}p\ \ ) = 2^1 = 2$, p any odd prime.

$f(q^s) = 2^1 = 2$, $q \equiv 1 \pmod 4$, q prime.

$f(p_1^{s_1}p_2^{s_2}) = 2^1$, not both $p_1, p_2 \equiv 1 \pmod 4$, p_1, p_2 odd primes.

$f(p_1^{s_1}p_2^{s_2}p_3^{s_3}) = 2^2 = 4$, p_1, p_2, p_3 odd primes not all contruent to 1 modulo 4.

$f(q^{t_1}q^{t_2}) = 2^2 = 4$, q_1, q_2 odd primes congruent to 1 modulo 4.

$f(2^{s+1}p^tq^r) = 2^2 = 4$, p, q any odd primes, etc.

We have $f(60) = f(4 \cdot 3 \cdot 5) = 2^2 = 4$. The corresponding PPTs are (899, 60, 901), (91, 60, 109), (11, 60, 61), (221, 60, 229). Also, $f(16) = f(2^4) = 1$. The corresponding PPT is (63, 16, 65).

We let the smallest integer n such that $f(n) = 2^k$ ($k = -\infty, 0, 1, 2, \ldots$) be denoted by $M(n, k)$.

It is easily seen that $M(n, -\infty) = 1$, $M(n, 0) = 3$, $M(n, 1) = 5$. An interesting result is stated in Theorem 3, but first we let $p_1, p_2, \ldots$ denote the successive odd primes and we denote the product of k successive odd primes by $\Pi_k = p_1p_2 \cdots p_k$.

Theorem 3

If $k \geq 2$ then $M(n, k) = 4\Pi_k$.

Proof: The reader can easily verify the relations

$$2^{s+1}O(k, 1) > O(k, 1) > 4\Pi_k,$$

$$2^{s+1}O(k, 3) > 4\Pi_k$$

and

$$O(k + 1, 3) > 4\Pi_k,$$

if $k \geq 2$, while all have the same value of $f(n) = 2^k$. This proves the theorem. We thus have $M(n, 2) = 4\Pi_2 = 60$, $M(n, 3) = 4 \cdot 3 \cdot 5 \cdot 7 = 420$, etc. Hence, 420 is the smallest number which appears exactly eight times in PPTs.

3. Perimeters

This is the most important part of our paper. It contains problems never investigated previously. To clarify them, we start with:

Definition 3

Let (x, y, z) be a PPT and (u, v) be its generator. We call the sum $x + y + z$ the PERIMETER of PPT.

We denote the perimeter of a PPT with generator (u, v) by

$$\Pi(u, v) = x + y + z = 2u(u + v) \equiv \Pi. \tag{3.1}$$

Thus $\Pi(2, 1) = 12$, $\Pi(3, 2) = 30$, $\Pi(4, 1) = 40$, etc. Different PPTs may have the same Π for different generators. An example of this will be given in Theorem 5. (No two different generators can lead to the same PPT.) Accordingly, we introduce:

Definition 4

The (exact) number of different PPTs having the same perimeter is called the DOMAIN of this perimeter. In symbols, we write $D(\Pi) = k$ if the number of generator pairs in the set $\{(u, v) | \Pi(u, v) = \Pi\}$ is k. Since a number n may not be a perimeter, we introduce the notation $n \neq \Pi$ and write $D(\Pi) = 0$.

By (3.1), every perimeter is even. Hence $D(2t + 1) = 0$. Let $m \equiv 1 \pmod 2$ and p be an odd prime such that $p^t > 2^s m$ for some s with $(p, m) = 1$. It is easy to prove that $D(2^s m p^t) = 0$. The method of proving this will emerge from the sequel.

Theorem 4

Let p be an odd prime.

a) If $2^{s+1}p = \Pi$ and $2^{s+1} > p > 2^s$, then $D(2^{s+1}p) = 1$.

(3.2) b) If $2p^t(p^t + 1) = \Pi$, then $D(2p^t(p^t + 2)) = 1$.

c) If $p^t(p^t + 1) = \Pi$, then $D(p^t(p^t + 1)) = 1$.

Proof: Generally, in order to investigate whether a given n is or is not a perimeter, it suffices to write n in the form $2u(u + v)$, where (u, v) is a generator. Then make use of the relation (1.2).

To prove [(3.2), a], we proceed as follows. Let $2^{s+1}p = 2u(u + v)$, then $2^s p = u(u + v)$. Since $u + v \equiv 1 \pmod 2$, we have $2^s | u$. There are therefore two cases $p | u$ or $p | (u + v)$. If $p | u$, we have $u = 2^s p$ and $u + v = 1$, which is impossible because $u + v < u$. If $p | (u + v)$, we have $u = 2^s$, $u + v = p$, and $v = p - 2^s$. By hypothesis, $u < u + v < 2u$. Obviously $(u, v) = 1$ and $u + v \equiv 1 \pmod 2$, so (u, v) is a generator and $D(2^{s+1}p) = 1$.

To prove [(3.2), b], we let $p^t(p^t + 2) = u(u + v)$. Since $p^t + 2$ may factor, we assume $p^t + 2 = fg$ with $f > g$. With $p^t fg = u(u + v)$, there are two obvious cases to consider. They are:

$$u = p^t,\ v = fg - p^t \quad \text{and} \quad u = fg,\ v = p^t - fg.$$

The latter case is out, since we need $v > 0$. The former case yields a solution since $(u, v) = 1$, $u + v \equiv 1 \pmod 2$, and $u < u + v < 2u$. With $(f, g) = 1$, $g \neq 1$, there are six more possibilities, all of which can be ruled out, since the relations

$$p^t fg < 1,\ p^t fg < 2,\ g < p^t f < 2g,\ p^t f < g < 2fp^t,$$

$$f < p^t g < 2f, \text{ and } gp^t < f < 2gp^t$$

are impossible. Therefore, $D(2p^t(p^t + 2)) = 1$.

An argument similar to that of [(3.2), b] will show that the only solution for [(3.2), c] is $u = (p^t + 1)/2$, $v = (p^t - 1)/2$, so that $D(p^t(p^t + 1)) = 1$, completing the proof of the theorem.

The following is an immediate consequence of Theorem 4.

Corollary

Let $2^{p-1}(2^p - 1)$, p prime, be a perfect number. Then $2^{p-1}(2^p - 1)$ cannot be a perimeter, while $2^p(2^p - 1)$ can be a perimeter only once.

Theorem 4 also shows that there are infinitely many PPTs of domain 1. We prove the following interesting result.

Theorem 5

Let be an odd prime.

(3.3)

a) When $p > 6$, $12p(p + 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv 1 \pmod 3$ and $D(\Pi) = 2$ if $p \equiv -1 \pmod 3$.

b) When $p > 8$, $12p(p - 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv -1 \pmod 3$ and $D(\Pi) = 2$ if $p \equiv 1 \pmod 3$.

Proof: Since $6p(p + 2) = u(u + v)$, where u is even, $u + v \equiv 1 \pmod 2$ and $(u, u + v) = 1$, we have eight possible cases for the choices of the factors of u and $u + v$. However, we need $u < u + v < 2u$, so six of these cases can be eliminated immediately leaving only

(3.4) $$u = 2p,\ v = 3p + 6 - 2p = p + 6$$

and

(3.5) $$u = 2(p + 1),\ v = p - 4.$$

When $p \equiv 1 \pmod 3$, then (3.5) is not a valid generator, since $(u, v) \neq 1$. However, (3.4) is a generator with perimeter $12p(p + 2)$. When $p \equiv -1 \pmod 3$ both (3.4) and (3.5) are valid generators of $12p(p + 2)$, since $(u, v) = 1$, $u + v \equiv 1 \pmod 2$, and $u < u + v < 2u$.

Let $6p(p - 2) = \Pi$. A similar argument to that of part (a) shows that

(3.6) $$u = 2p,\ v = p - 6$$

and

(3.7) $$u = 2(p - 2),\ v = p + 4$$

are generators of $12p(p - 2)$ if $p \equiv 1 \pmod 3$, while only (3.6) is a valid generator if $p \equiv -1 \pmod 3$.

By Dirichlet's theorem and Theorem 5, we know there are infinitely many PPTs with $D(\Pi) = 2$.

Actually, Theorem 5 is a special case of the following more general theorem whose proof we omit because of its similarity to that of Theorem 5.

Theorem 5a

Let p be an odd prime. Let q be a prime such that $2^q - 1$ is a prime.

(i) When $p > 2(2^q - 1)$, $2^q(2^q - 1)p(p + 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv -2 \pmod{2^q - 1}$ and $D(\Pi) = 2$ if $p \not\equiv -2 \pmod{2^q - 1}$. The solutions are

$$u = 2^{q-1}p,\ v = (2^{q-1} - 1)p + 2(2^q - 1)$$

and

$$u = 2^{q-1}(p + 2),\ v = (2^{q-1} - 1)p - 2^q$$

if $p \not\equiv -2 \pmod{2^q - 1}$. If $p \equiv 2 \pmod{2^q - 1}$, only the first solution is a valid generator.

(ii) When $p > 2^{q+1}$, $2^q(2^q - 1)p(p - 2) = \Pi$ has $D(\Pi) = 1$ if $p \equiv 2 \pmod{2^q - 1}$ and $D(\Pi) = 2$ if $p \not\equiv 2 \pmod{2^q - 1}$. The solutions are

$$u = 2^{q-1}p,\ v = (2^{q-1} - 1)p - 2(2^q - 1)$$

and

$$u = 2^{q-1}(p - 2),\ v = (2^{q-1} - 1)p + 2^q$$

if $p \not\equiv 2 \pmod{2^q - 1}$. If $p \equiv 2 \pmod{2^q - 1}$, only the first solution is a valid generator.

When $p \equiv -1 \pmod 3$ and $p + 2$ is also a prime, the two solutions of parts (a) and (b) of Theorem 5 are the same. Hence, twin primes enter into our analysis of the perimeter problem.

It is easy to show that the smallest value of Π with $D(\Pi) = 2$ is

$$12 \cdot 11 \cdot 13 = 1716.$$

The generators are $\Pi = \Pi(22, 17) = \Pi(26, 7)$, whose Pythagorean triples are, respectively, (195, 748, 773) and (627, 364, 725).

4. More on Domains

The following two theorems state the most important results of this paper. In the sequel, it will be convenient to denote the two numbers $T = p^s$ and $T + 2 = q^t$, where p, q are odd primes, by *prime power twins*. We state:

Theorem 6

Let

$$\Pi = 2u(u + v),\ T \text{ and } T + 2 \text{ be prime power twins,}$$
$$T > \Pi,\ D(\Pi) = k, \text{ and } (\Pi, T(T + 2)) = 1. \tag{4.1}$$

Then

$$\Pi' = \Pi T(T + 2) \text{ is a perimeter with } D(\Pi') = 2k. \tag{4.2}$$

Proof: We prove that any generator for Π leads to exactly two generators for Π'. Since $T > \Pi > 2(u + v)$, we see that

$$(4.3) \qquad T > 2(u + v)/(u - v) \quad \text{and} \quad T > 2u/v.$$

But, (4.3) implies that $T(u - v)/(u + v) > 2$, so $2uT/(u + v) > (T + 2)$ or

$$(4.4) \qquad 2uT > (u + v)(T + 2) > uT.$$

Furthermore, from (4.3) we obtain $2/T < v/u$, so

$$(T + 2)/T < (u + v)/u \quad \text{or} \quad (u + v)T > u(T + 2).$$

Hence, by (1.2),

$$(4.5) \qquad 2u(T + 1) > (u + v)T > u(T + 2).$$

Since we want $\Pi' = \Pi T(T + 2) = 2u(u + v)T(T + 2) = 2x(x + y)$, where (x, y) is a generator, there are sixteen possible ways of choosing the factors of $u(u + v)T(T + 2)$ for x and $x + y$. However, we need $x < x + y < 2x$. Therefore, fourteen of these possibilities can be easily eliminated. For example, if $x = u(u + v)$ and $x + y = T(T + 2)$, then

$$T(T + 2) > 2(u + v)(T + 2) > 2(u + v)T > 4(u + v)^2 > 2u(u + v),$$

so $x + y > 2x$. As another example, let $x = T(T + 2)$ and $x + y = u(u + v)$. Then

$$T(T + 2) > u(u + v),$$

so $x > x + y$. The only two cases that satisfy $x < x + y < 2x$, by (4.4) and (4.5), are

$$(4.6) \qquad x = uT,\ x + y = (u + v)(T + 2),\ y = (u + v)(T + 2) - uT$$

and

$$(4.7) \qquad x = u(T + 2),\ x + y = (u + v)T,\ y = vT - 2u.$$

In both of these cases, it is easy to show that $(x, y) = 1$, $x + y \equiv 1 \pmod 2$ and $2x(x + y) = \Pi'$.

Because $u(u + v) = f \cdot g$ with $(f, g) = 1$, where $f > g$ is possible, since $u(u + v) = p^s$, p a prime, is impossible, we need to show that these factorizations do not lead to any new generators of Π'. We let $f > 2g$, then $2g > f > g$ and $2f > g > f$ are both impossible, so that (f, g) is not a generator of Π.

With $2u(u + v)T(T + 2) = 2fgT(T + 2) = 2x(x + y)$, where (x, y) is a generator, there are, again, sixteen possible ways of choosing the factors of $fgT(T + 2)$ for x and $x + y$. All of these cases are easily eliminated. For example, if $x + y = g(T + 2)$ and $x = fT$, then $fT > 2gt > g(T + 2)$, so $x > x + y$, which contradicts (x, y) being a generator; as another example, let $x + y = gT$ and $x = f(T + 2)$, then $f(T + 2) > 2g(T + 2) > gT$ and again $x > x + y$, which is a contradiction. As our final example, we choose $x + y = fT$ and $x = g(T + 2)$. Then $4g > (f - 2g)T > T > 2u(u + v) = 2fg$, so that $2 > f$, which is a contradiction. We leave the other cases to the reader.

Hence, we have proved that the only generators for Π' are the generators for Π, each of which leads to exactly two generators for Π'. This proves the theorem.

Example

Let $\Pi = \Pi(22, 17) = \Pi(26, 7) = 1716$. We choose $T = 1721$ and $T + 2 = 1723$, where 1721 and 1723 are primes with $T > \Pi$. Hence,

$$\Pi' = 1716 \cdot 1721 \cdot 1723,\ D(\Pi') = 4,$$

and

$$\Pi' = \Pi'(37862, 29335) = \Pi'(37906, 29213)$$

$$= \Pi'(44746, 12113) = \Pi'(44798, 11995).$$

With doubling the $D(\Pi)$, the PPTs grow enormously, since $T > \Pi$. Hence, if the T's are finite in number, there may be an upper bound for $D(\Pi)$. The following modification of Theorem 6 may somehow be helpful.

Theorem 6a

Let

$$\Pi = 2u(u + v),\ (T,\ T + 2) \text{ be prime power twins, } D(\Pi) = k,$$
$$(\Pi,\ T(T + 2)) = 1, \text{ and } T > 2(u + v).$$

(4.8) Let the number of pairs $(f, g) = 1$, such that

$$u(u + v) = f \cdot g,\ f > 2g, \text{ and } 2g(T + 2) > fT > g(T + 2)$$

with f odd be m, where $m = 0, 1, 2, \ldots$.

Then $D(\Pi') = D(\Pi T(T + 2)) = 2k + m$.

Proof: With $T > 2(u + v) \geq 2(u + v)/(u - v)$ and $T > 2(u + v) > 2u/v$, we prove, as in Theorem 6, that each generator for Π leads to exactly two generators of $\Pi' = 2u(u + v)T(T + 2)$. Since $2g(T + 2) > fT > g(T + 2)$, (4.5) would account for another solution, so $D(\Pi') = 2k + m$, $m \geq 0$. The author was unable to find an example where $m \neq 0$.

Example

Let $\Pi = \Pi(22, 17) = \Pi(26, 7) = 1716$, then $2(u + v)$ equals 78 or 66. For $T > 78$, we choose $T = 101$ and $T + 2 = 103$. We then have

$$g \cdot f = 6(11 \cdot 13) = 2(3 \cdot 11 \cdot 13)$$

with $2g < f$ and f odd. But in neither of these cases does the relation

$$2g(T + 2) > fT > g(T + 2)$$

hold, as can be easily verified. When $T > 66$, we choose $T = 71$ and $T + 2 = 73$. Then

$$g \cdot f = 6(11 \cdot 13) = (3 \cdot 11 \cdot 13).$$

Again, in neither case, is

$$2g(T + 2) > fT > g(T + 2).$$

Thus, $m = 0$ and $\Pi' = 1716 \cdot 101 \cdot 103$ has $D(\Pi') = 4$.

Theorem 7

Let $\Pi = 2u(u + v)$. Let $(T, T + 2)$ be prime power twins with

$$(\Pi, T(T + 2)) = 1.$$

Let $D(\Pi) = k$. Further, let

(4.9)
$$\left(\frac{\Pi}{2} + 1\right)^{1/2} - 1 < T < (\Pi + 1)^{1/2} - 1,\ u - v \geq 5, \text{ and } v \geq 3, \text{ or}$$
$$\left(\frac{\Pi}{4} + 1\right)^{1/2} - 1 < T < \left(\frac{\Pi}{2} + 1\right)^{1/2} - 1,\ u - v \geq 6, \text{ and } v \geq 5 \text{ with } u \text{ odd.}$$

Then

(4.10)
$$\Pi' = 2u(u + v)T(T + 2) \text{ has } D(\Pi') = 2k + 1 + m,$$
$$m \text{ as in Theorem 6a.}$$

Proof: With the restrictions on $u - v$, and v from (4.9), we can easily prove that

$$\left(\frac{\Pi}{2} + 1\right)^{1/2} - 1 > 2(u + v)/(u - v) \quad\text{and}\quad \left(\frac{\Pi}{2} + 1\right)^{1/2} - 1 > 2u/v.$$

Also

$$\left(\frac{\Pi}{4} + 1\right)^{1/2} - 1 > 2(u + v)/(u - v) \quad\text{and}\quad \left(\frac{\Pi}{4} + 1\right)^{1/2} - 1 > 2u/v.$$

Thus, $T > 2(u + v)/(u - v)$ and $T > 2u/v$. From these last two relations, it is then proved, as before, that every generator (u, v) for Π leads to exactly two generators for $\Pi' = 2u(u + v)T(T + 2)$. We further have, from part (b) of (4.9), that

$$2T(T + 1) > u(u + v) > T(T + 2),$$

and from part (a) of (4.9) that

$$2u(u + v) > T(T + 2) > u(u + v) \text{ for a fixed } T.$$

This would account for the additional generator for Π'. The meaning of the possible m generators for Π' is the same as in Theorem 6a. This completes the proof of Theorem 7.

The reader may ask whether in the intervals given by (4.9) there is always a prime power (or prime) T. This fundamental question is answered affirmatively by a famous theorem by Chebyshev [1] which states that in the interval $(y, (1 + e)y)$, $e > 1/5$ there is always, from a certain point on, one prime, from a further point on, two primes, etc. The reader will easily verify that the intervals (4.9) satisfy the conditions of Chebyshev's theorem.

As our first example, we choose $\Pi = \Pi(40, 3) = 3440$, so that $\sqrt{1721} - 1 < T < \sqrt{3440} - 1$ or $40 < T < 57$. With $T = 41$ and $T + 2 = 43$, we have

$$(\Pi, T(T + 2)) = 43,$$

so that Theorems 6, 6a, and 7 do not apply. We choose $T = 47$ and $T + 2 = 7^2$. Note that $D(\Pi) = 1$ and $\Pi' = 3440 \cdot 47 \cdot 49 = 7922320$. Since $1720 = 8(5 \cdot 43) = gf$ with $f > 2g$ and f odd does not yield

$$2g(T + 2) > fT > g(T + 2),$$

we have $m = 0$ and $D(\Pi') = 3$.

As another example, we choose $\Pi(46, 29) = \Pi(50, 19) = 4 \cdot 3 \cdot 23 \cdot 25 = 6900$ so that $D(\Pi) = 2 = k$. We have $u - v = 17$, $v = 29$ and $u - v = 31$, $v = 19$. Further

$$\sqrt{\frac{1}{2} \cdot 6900 + 1} - 1 < T < \sqrt{6900 + 1} - 1,$$

and we choose $T = 59$, $T + 2 = 61$, so that $\Pi' = 6900 \cdot 59 \cdot 61 = 24833100$. We also have $u(u + v) = 6(23 \cdot 25) = 2(3 \cdot 23 \cdot 25) = fg$ with $f > 2$ and f odd. But the condition

$$2g(T + 2) > fT > g(T + 2)$$

is not satisfied here. Thus, by Theorem 7, $D(\Pi') = D(24833100) = 5$. The author leaves it to the reader to find the value of $D(\Pi')$ when $T = 71$, $T + 2 = 73$ and $T = 79$, $T + 2 = 3^4$.

5. n-Periadic Numbers

We introduce

Definition 5

A number t is called n-PERIADIC if t^n is a perimeter but t^{n+1} is not.

If t^n is a perimeter, then there exist x and y relatively prime such that $x + y \equiv 1 \pmod 2$, $2x(x + y) = t^n$, and $x < x + t < 2x$. Hence, there exist u and v relatively prime such that $x = 2^{n-1}u^n$ and $x + y = v^n$. Furthermore, $2^{(n-1)/n}u < v < 2u$. If t^{n+1} is not a perimeter, then $v < 2^{n/n+1}u$. This proves the necessary part of the following theorem.

Theorem 8

The number t is n-periodic iff there exists $(u, v) = 1$ such that

(5.1) $$2^{(n-1)/n}u < v < 2u \quad \text{and} \quad v < 2^{n/(n+1)}u.$$

We leave a proof of the sufficiency part to the reader.

From (5.1), we see that $v > 2^{(n-1)/n}u > \left(1 + \frac{n-1}{n}\ln 2\right)u$, so

(5.2) $$\frac{n-1}{n} < \frac{v}{u} < 2.$$

When $n = 2$, (5.2) yields $\frac{1}{2} < \frac{v}{u} < 2$. Choose $v = 6s + 1$ and $u = 4s + 1$ with $s \geq 2$. Then $(u, v) = 1$ and $v^2 > 2u^2$. Furthermore, $v^3 < 4u^3$. Let

$$x = 2(4s + 1)^2 \quad \text{and} \quad x + y = (6x + 1)^2,$$

as in the proof of the theorem. Then

(5.3) $$t = \Pi(4s + 1, 2s)$$

is 2-periodic. In particular, with $s = 2$, we have that $\Pi(9, 4) = 18 \cdot 13$ is 2-periodic with generator $x = 162$, $y = 7$.

When $n = 3$, (5.2) yields $\frac{2}{3} < \frac{v}{u} < 2$. Choose $v = 10s + 1$ and $u = 6s + 1$ with $s \geq 2$. Then $(u, v) = 1$, $v^3 > 4u^3$ and $v^4 < 8u^4$. Let

$$x = 4(6s + 1)^3 \quad \text{and} \quad y + x = (10s + 1)^3.$$

Then

(5.4) $$t = \Pi(6s + 1, 4s)$$

is 3-periodic. In particular, for $s = 2$, we have

$$v = 21,\ u = 13,\ x = 4 \cdot 13^3 = 8788,\ y = 473$$

and $t = \Pi(13, 8)$ is 3-periodic.

By this method, we can obtain any n-periodic number. However, those obtained by (5.3) and (5.4) are by far not all of the infinitely many 2-periodic and 3-periodic numbers.

Conspicuously absent are the 1-periodic numbers. We have,

(5.5) $$\Pi = \Pi(u, 1),\ u \geq 3$$

is 1-periodic, since $2u > u + 1 > u$ and $(u + 1)^2 > 2u^2$.

The reader should not overlook the following trivial relation. Let $\Pi(u, v) = 2u(u + v)$, then

(5.6) $$(\Pi(u, v))^n = \Pi(2^{n-1}u^n, (u + v)^n - 2^{n-1}u^n),\ n \geq 1.$$

Note that if $\Pi(u, v) = x + y + z$, then

$$\Pi(u, v)^2 = x\Pi(u, v) + y\Pi(u, v) + z\Pi(u, v),$$

but

$$(\Pi(u, v)x, \Pi(u, v)y, \Pi(u, v)z) \neq \Pi(u, v).$$

In this context, we prove

Theorem 9

For every perimeter $\Pi(u, v)$ there exists at least one prime p such that $p\Pi(u, v)$ is a perimeter.

Proof: Let $\Pi = 2u(u + v)$. By Bertrand's postulate, there is at least one prime p such that $2u(u + v) > p > u(u + v)$. Hence, $2u(u + v)p$ is a perimeter.

6. Associating with Fibonacci

We introduce

Definition 6

Let (x, y, z) be a PPT. It is called *associative* if $f(x) = f(y) = f(z)$, *nonassociative* if all PFIs of x, y, z are different, *quasi-associative* if the PFIs of exactly any two x, y, z are equal. If

$$f(x) = f(y) = f(z) = 2^k, \; k = 0, 1, \ldots,$$

we say the PPT = (x, y, z) is *k-associative*.

Examples

(3, 4, 5) is quasi-associative,

(5, 12, 13) is 1-associative,

(7, 24, 25) is quasi-associative,

(99, 100, 101) is 1-associative, since $f(99) = f(3^2 \cdot 11) = 2^1$, $f(100) = f(4 \cdot 5^2) = 2^1$ and $f(101) = 2^1$,

(675, 52, 677) is quasi-associative,

(11, 60, 61) is nonassociative, since $f(11) = 2^0$, $f(60) = f(4 \cdot 3 \cdot 5) = 2^2$, $f(61) = 2^1$,

(3477, 236, 3485) is nonassociative, since $f(3477) = f(3 \cdot 19 \cdot 61) = 2^2$, $f(236) = f(4 \cdot 59) = 2^1$, $f(3485) = f(5 \cdot 17 \cdot 41) = 2^3$.

The Fibonacci sequence

$$F_1 = F_2 = 1,\ F_{n+2} = F_n + F_{n+1} \quad (n = 1,\ 2,\ \ldots),$$

has solved and raised many puzzles. Every mathematician should have a copy of Hoggatt's precious booklet [3] on this subject. Since $F_{6k+3} \equiv 2 \pmod 4$, F_{6k+3} does not appear in any PPT; all other F_n, $n > 3$, do. $F_{12} = 144$ has $\Pi = \Pi(8, 1)$, with the PPT being (63, 16, 65). The only Fibonacci numbers known to appear in the same PPT are 3, 5 and 5, 13, see [5]. The Fibonacci number $F_8 = 21$ has (21) = 2 where the two PPTs are (21, 20, 29) and (21, 220, 221). Note that (21, 220, 221) is quasi-associative, since

$$f(21) = f(3 \cdot 7) = 2^1,\ f(220) = f(4 \cdot 5 \cdot 11) = 2^2,$$

and

$$f(221) = f(13 \cdot 17) = 2^2.$$

Observe that (21, 20, 29) is also quasi-associative. The Fibonacci number $F_{11} = 89$ appears in (89, 3960, 3961) with

$$f(89) = 2^1,\ f(3960) = f(8 \cdot 3^2 \cdot 5 \cdot 11) = 2^3,$$

and

$$f(3961) = f(17 \cdot 233) = 2^2.$$

Hence, the triple is nonassociative. The first Fibonacci number which is a perimeter is 144, the largest perfect square in the Fibonacci sequence. $\Pi(8, 1) = 144$ leads to the PPT (63, 16, 65), with $D(144) = 1$. This PPT is nonassociative with

$$f(63) = 2^1,\ f(16) = 2^0,\ f(65) = 2^2.$$

Concluding, we want to point out that apart from the riddle of associativity the most saddening unsolved problem in this paper is the question of whether or not there are infinitely many PPTs of any given domain. Since a solution seems to hinge on the unsolved problem of the number of prime twins, it seems to be a difficult problem.

Acknowledgment

The author is deeply indebted to the editor for improving important results and adding new ones, for refining concepts and definitions, and for correcting proofs and calculating errors. Without his contribution, this paper would not be complete.

References

1. Chebyshev. See Edmund Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, I. New York: Chelsea, 1953.
2. Leonard E. Dickson. *History of Number Theory*. 3 vols. New York: Chelsea, 1919.
3. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton-Mifflin, 1969 (rpt. The Fibonacci Association, 1980).

4. Edmund Landau. *Vorlesungen uber Zahlentheorie*. New York: Chelsea, 1950.
5. Marjorie Bicknell-Johnson. "Pythagorean Triples Containing Fibonacci Numbers: Solutions for $F_n^2 \pm F_k^2 = K^2$." *The Fibonacci Quarterly* 17, no. 1 (1979):1-12.
6. Marjorie Bicknell-Johnson. "Addend to 'Pythagorean Triples Containing Fibonacci Numbers: Solutions for $F_n^2 \pm F_k^2 = K^2$.'" *The Fibonacci Quarterly* 17, no. 4 (1979):293.

*****