CHARACTERIZATION OF A SEQUENCE

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In [1], Hoggatt and Johnson characterize all integral sequences $\{u_n\}$ satisfying

(1) $$u_{n+1}u_{n-1} - u_n^2 = (-1)^n.$$

The purpose of this paper is to characterize all sequences which satisfy the relation

(2) $$s_n^2 - s_m^2 = s_{n+m}s_{n-m}$$

for all integers m and n. Of necessity, we see that

(3) $$s_0 = 0,$$

while $m = -n$ yields

(4) $$s_{-n} = \pm s_n$$

for all integers n. Let $n = 0$ in (2), then replace m by n. This gives

(5) $$s_n(s_n + s_{-n}) = 0$$

for all integers n. Replacing n by $n + 1$ and m by n in (2) yields

(6) $$s_{n+1}^2 - s_n^2 = s_{2n+1}s_1$$

for all integers n.

Letting $s_1 = 0$ in (6) and using mathematical inducation with (6) we see that $s_n = 0$ for all nonnegative integers. However, by (4) we than have $s_n = 0$ for all integers n. The sequence, all of whose terms are 0, obviously satisfies (2), so for the remainder of this paper we assume $s_1 = a \neq 0$. By (5), we than have

(7) $$s_{-n} = -s_n.$$

Using (2) with $n = 2k + 1$, $m = 2k - 1$ and $n = 2k + 2$, $m = 2k$, we obtain $s_{2k+1}^2 - s_{2k-1}^2 = s_{4k}s_2$ and $s_{2k+2}^2 - s_{2k}^2 = s_{4k+2}s_2$, so that when $s_2 = 0$ we have

(8) $$s_{2k+1} = \pm s_{2k-1}$$

and

$$s_{2k+2} = \pm s_{2k}. \tag{9}$$

Mathematical induction and (9) together with (7) imply that $s_{2n} = 0$ for all integers n. Furthermore, (8) and mathematical induction together with (7) tell us that $s_{2n+1} = \pm a$ for all integers . However, $s_{2n}^2 - s_1^2 = s_{2n+1}s_{2n-1}$, so $-s_1^2 = s_{2n+1}s_{2n-1}$ showing that s_{2n+1} and s_{2n-1} have opposite signs. Therefore, with $s_1 = a \neq 0$ and $s_2 = 0$, we have

$$s_n = \begin{cases} a, & n \equiv 1 \pmod 4 \\ -a, & n \equiv -1 \pmod 4 \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

The sequence just calculated in (10) is a solution to the problem at hand because, if n and m are of the same parity, then $m + n$ and $m - n$ are even, and $s_n^2 = s_m^2$ so $s_n^2 - s_m^2 = 0 = s_{n+m}s_{n-m}$. If n is odd and m is even, then $n + m$ and $n - m$ are odd and separated by $2m$, which is a multiple of 4. Hence,

$$s_{n+m}s_{n-m} = a^2 = s_n^2 - s_m^2.$$

Similarly, if n is even and m is odd.

Throughout the remainder of this paper, we assume that $s_2 = b \neq 0$ and $s_1 = a \neq 0$. From (6) and (2),

$$as_{2n+1} = s_{n+1}^2 - s_n^2 = (s_{n+1}^2 - s_{n-1}^2) - (s_n^2 - s_{n-1}^2) = bs_{2n} - as_{2n-1},$$

so that

$$s_{2n+1} = \frac{bs_{2n} - as_{2n-1}}{a}, \text{ for all } n \tag{11}$$

or, equivalently,

$$a(s_{2n+1} + s_{2n-1}) = bs_{2n}. \tag{12}$$

Now

$$\begin{aligned} s_{n+2}^2 - s_n^2 = bs_{2n+2} &= (s_{n+2}^2 - s_{n-1}^2) + (s_{n-1}^2 - s_n^2) \\ &= s_3 s_{2n+1} + s_{2n-1}s_{-1}. \end{aligned}$$

Furthermore, by (11), $s_3 = (b^2 - a^2)/a$ and by (7), $s_{-1} = -a$. Hence, substitution and (12) yield

$$\begin{aligned} bs_{2n+2} &= \frac{b^2 - a^2}{a} s_{2n+1} - as_{2n-1} \\ &= \frac{b^2}{a} s_{2n+1} - a(s_{2n+1} + s_{2n-1}) \\ &= \frac{b^2}{a} s_{2n+1} - bs_{2n}. \end{aligned} \tag{13}$$

Hence,

(14) $$s_{2n+2} = \frac{bs_{2n+1} - as_{2n}}{a}, \text{ for all } n.$$

Combining (11) and (14), we have

(15) $$s_{k+1} = \frac{bs_k - as_{k-1}}{a}, \text{ for all } k.$$

Therefore, the only sequences other than the two exceptions which might satisfy (2) for all n and m must be second-order linear recurrences of the form (15), where $s_1 = a \neq 0$ and $s_2 = b \neq 0$.

Using standard techniques with

$$\alpha = \frac{b + \sqrt{b^2 - 4a^2}}{2a} \quad \text{and} \quad \beta = \frac{b - \sqrt{b^2 - 4a^2}}{2a}$$

as the roots of $ax^2 - bx + a = 0$, we see that, for all integers n,

(16) $$s_n = \begin{cases} \dfrac{a(\alpha^n - \beta^n)}{\alpha - \beta}, & b \neq \pm 2a \\ na, & b = 2a \\ (-1)^{n+1}na, & b = -2a \end{cases}$$

If $s_n = na$ or $s_n = (-1)^{n+1}na$ for all n, then it is easy to verify the truth of (2). Hence, we assume $b \neq \pm 2a$; then, with $\alpha\beta = 1$, we have

$$s_n^2 - s_m^2 = \left(\frac{a}{\alpha - \beta}\right)^2 [(\alpha^{2n} - 2 + \beta^{2n}) - (\alpha^{2m} - 2 + \beta^{2m})]$$
$$= \left(\frac{a}{\alpha - \beta}\right)^2 (\alpha^{2n} - \alpha^{2m} + \beta^{2n} - \beta^{2m}).$$

Furthermore,

$$s_{n+m}s_{n-m} = \left(\frac{a}{\alpha - \beta}\right)^2 (\alpha^{2n} - \alpha^{2m} + \beta^{2n} - \beta^{2m}),$$

and again (2) is true for all integers n and m. Thus, we have found all sequences satisfying (2) for all integers n and m.

It is interesting to note that the Fibonacci and Lucas sequences do not satisfy (15). However, the sequence of Fibonacci numbers $\{F_{2n}\}_{n=1}^{\infty}$ does, if we let $a = 1$ and $b = 3$, for then

$$\alpha = \frac{3 + \sqrt{5}}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^2, \ \beta = \left(\frac{1 - \sqrt{5}}{2}\right)^2, \text{ and } s_n = F_{2n}.$$

Another interesting example of such a sequence is found by letting $s_1 = 1$ and $s_2 = i$, then

$$s_3 = -2,\ s_4 = -3i,\ s_5 = 5,\ s_6 = 8i,\ s_7 = -13,\ s_8 = -21i, \text{ etc.}$$

It should be noted that s_n is an integer for all integers n if and only if a and b are integers and a divides b. This follows directly by using the recursive formula in the form

$$s_{n+1} = \frac{b}{a} s_n - s_{n-1},$$

for then, by induction,

$$s_k = \frac{b^{k-1}}{a^{k-2}} + (\text{integer}) \frac{b^{k-3}}{a^{k-4}} + \cdots + (\text{integer}) \frac{b^3}{a^2} + (\text{integer})\, b, \ k \text{ even}$$

and

$$s_k = \frac{b^{k-1}}{a^{k-2}} + (\text{integer}) \frac{b^{k-3}}{a^{k-4}} + \cdots + (\text{integer}) \frac{b^2}{a} + (\text{integer})\, a, \ k \text{ odd.}$$

Hence, by induction, $s_n \in Z$ if and only if a^n divides b^{n+1} for all $n \geq 3$, but then a must divide b.

Also note that if a divides b, then a divides s_n for all integers n. Hence, the only integral solutions to the problem are multiples of those generated by letting $s_1 = 1$ and $s_2 = b$, where b is an integer.

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Reference

V. E. Hoggatt, Jr. and Marjorie Bicknell Johnson. "A Primer for the Fibonacci Numbers XVII: Generalized Fibonacci Numbers Satisfying $u_{n+1}u_{n-1} - u_n^2 = \pm 1$." *The Fibonacci Quarterly* 16, no. 2 (1978):130-37.

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