A NOTE ON THE FAREY-FIBONACCI SEQUENCE

(Submitted October 1980)

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1. Introduction

The Fibonacci sequence $\{F_n : n \geq 0\}$ is defined as

$$F_0 = 1,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

Let $r_{i,j} = F_i/F_j$. Alladi [1] defined a Farey-Fibonacci sequence f_n of order n as the sequence obtained by arranging the terms of the set

$$\sum_n = \{r_{i,j} \mid 1 \leq i < j \leq n\}$$

in ascending order and studied its properties in detail. Alladi [2] and Gupta [3] gave rapid methods to write out f_n. Finally, Alladi and Shannon [4] briefly considered certain special properties of consecutive members of f_n.

We now prescribe a different scheme to write out f_n, which is rapid, direct, and simpler than the earlier approaches. We not only obtain the term-number of a preassigned member of f_n as found by Gupta [3], but also a formula for the general term of f_n not explicitly obtained before.

2. Scheme

Let us write out the terms of $\sum_n$ in a triangular array as shown below:

$$\begin{array}{l}
r_{1,n};\ r_{1,n-1};\ r_{1,n-2};\ \ldots;\ r_{1,1+n-i};\ \ldots;\ r_{1,2} \\
\qquad r_{2,n};\ r_{2,n-1};\ \ldots;\ r_{2,2+n-i};\ \ldots;\ r_{2,3} \\
\qquad\qquad r_{3,n};\ \ldots;\ r_{3,3+n-i};\ \ldots;\ r_{3,4} \\
\qquad\qquad\qquad \ldots\ldots\ldots\ldots\ldots\ldots \\
\qquad\qquad\qquad\qquad r_{i,n};\ \ldots;\ r_{i,i+1} \\
\qquad\qquad\qquad\qquad\qquad \ldots\ldots \\
\qquad\qquad\qquad\qquad\qquad\quad r_{n-1,n}
\end{array}$$

Next, we designate the terms of the ith column of this array by

$$x_1,\ x_2,\ \ldots,\ x_i.$$

Clearly, $x_j = r_{j,\,j+n-i}$ for $1 \leq j \leq i$. Observe that

(i) $x_1 < x_2$, an inequality equivalent to $F_{n-i} < F_{1+n-i}$

and

(ii) x_k lies between x_{k-1} and x_{k-2} for $3 \leq k \leq i$,

a consequence of the simple rule that the fraction

$$(h + h')/(k + k')$$

lies between h/k and h'/k'.

Let $a_{i,1};\ a_{i,2};\ \ldots;\ a_{i,i}$ denote the sequence obtained by arranging the x's in ascending order. Then the observations (i) and (ii) above imply

$$\text{(A)}\qquad \begin{aligned} &a_{i,1} = x_1;\ a_{i,i} = x_2\\ &a_{i,2} = x_3;\ a_{i,i-1} = x_4\\ &\text{and so on.}\end{aligned}$$

In fact, the x's arranged in ascending order are

$$x_1,\ x_3,\ x_5,\ \ldots,\ x_6,\ x_4,\ x_2.$$

This reveals the scheme of writing, in ascending order, the members of any given column of the above array.

Now since $a_{i,i} < a_{i+1,1}$ for $1 \leq i \leq n-1$ is equivalent to $F_{n-i} < F_{1+n-i}$ for $1 \leq i \leq n-1$, we get f_n as follows:

$$a_{1,1};\ a_{2,1};\ a_{2,2};\ \ldots;\ a_{i,1};\ a_{i,2};\ \ldots$$

$$\ldots\ a_{i,i};\ a_{i+1,1};\ a_{i+2,2};\ \ldots,\ a_{i+1,i+1};\ \ldots;\ a_{n-1,1};\ \ldots;\ a_{n-1,n-1}.$$

3. Formulas

I. If F_q/F_m is the tth term (T_t) of f_n, then

$$t = \begin{cases} \frac{1}{2}(n - m + q)(n - m + q - 1) + \frac{q+1}{2}: & \text{if } q \text{ is odd,}\\[2ex] \frac{1}{2}(n - m + q)(n - m + q - 1) + n - m + \frac{q}{2} + 1: & \text{if } q \text{ is even.}\end{cases}$$

Proof: If F_q/F_m or $r_{q,m}$ appears in the ith column of the array, then obviously $m - q = n - i$, $t = \frac{1}{2}i(i - 1) + j$, and from (A) $j = M$ or $i - M + 1$ according as $q = 2M - 1$ or $2M$, respectively. Thus t is apparent.

II. The following is the formula for the tth term of f_n:

$$T_t = F_{i-2|k|+\delta(i,k)}/F_{n-2|k|+\delta(i,k)},$$

where

$$i = \begin{cases} [\sqrt{(2t-2)}] & \text{if } 2t \leq [\sqrt{(2t-2)}]([\sqrt{(2t-2)}]+1) \\ [\sqrt{(2t-2)}]+1 & \text{otherwise,} \end{cases}$$

$$k = t - i(i-1)/2 - [(i+1)/2],$$

and

$$\delta(i, k) = \begin{cases} -1 \text{ if } i \text{ is even and } k \leq 0 \\ 0 \text{ if } i \text{ is odd and } k \leq 0 \\ 1 \text{ if } i \text{ is odd and } k > 0 \\ 2 \text{ if } i \text{ is even and } k > 0. \end{cases}$$

Proof: If T_t appears in the ith column of the array, then

$$i(i-1)/2 + 1 \leq t \leq (i+1)_i/2$$

and consequently i is as described above. Furthermore, if

$$T_t = a_{i,j} = x_p = r_{p,p+n-i},$$

then

$$i(i-1)/2 + j = t.$$

To find p, we examine its dependence on k where $j = [(i+1)/2] + k$. From relations (A) it is clear that

$$\text{for even } i,\ p = \begin{cases} i + 2k - 1 & \text{if } k \leq 0 \\ i - 2k + 2 & \text{if } k > 0 \end{cases}$$

and

$$\text{for odd } i,\ p = \begin{cases} i + 2k & \text{if } k \leq 0 \\ i - 2k + 1 & \text{if } k > 0. \end{cases}$$

These observations suffice.

References

1. K. Alladi. "A Farey Sequence of Fibonacci Numbers." *The Fibonacci Quarterly* 13, no. 1 (1975):1-10.
2. K. Alladi. "A Rapid Method to Form Farey-Fibonacci Fractions." *The Fibonacci Quarterly* 13, no. 1 (1975):31-32.
3. H. Gupta. "A Direct Method of Obtaining Farey-Fibonacci Sequences." *The Fibonacci Quarterly* 14, no. 4 (1976):389-91.
4. A. G. Shannon and K. Alladi. "On a Property of Consecutive Farey-Fibonacci Fractions." *The Fibonacci Quarterly* 15, no. 2 (1977):153-55.

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