ON A SYSTEM OF DIOPHANTINE EQUATIONS CONCERNING THE POLYGONAL NUMBERS

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1. INTRODUCTION

For the integer k ($k \geqslant 3$) and the natural number n, we call the integer

$$P_{n,k} = \frac{1}{2}\{(k-2)n^2 - (k-4)n\}$$

the nth polygonal number of order k. If $k = 3$, this number is called the nth triangular number and is denoted by t_n.

Wieckowski [1] showed that the system of Diophantine equations

$$\begin{aligned} t_x + t_y &= t_u \\ t_x + t_z &= t_v \\ t_y + t_z &= t_w \end{aligned}$$

has infinitely many solutions. It seems difficult to establish the counterpart of this theorem for general polygonal numbers.

In this paper it will be shown that the system of Diophantine equations

$$\begin{aligned} P_{x,k} + P_{y,k} &= P_{u,k} \\ P_{x,k} + P_{z,k} &= P_{v,k} \end{aligned}$$

has infinitely many solutions for any integer k ($k \geqslant 3$). In other words, there are infinitely many polygonal numbers of order k which can be represented in two different ways as the difference of polygonal numbers of order k.

To show this, we establish a stronger theorem in a manner similar to that used earlier in [2].

THEOREM: Let a and b be integers such that $a > 0$ and $a \equiv b \pmod 2$, and let

$$A_n = \frac{1}{2}(an^2 + bn) \qquad (n = 1, 2, 3, \ldots).$$

There are an infinite number of A_n's which can be expressed in two different ways as the difference of numbers of the same type.

2. PROOF OF THE THEOREM

First, we prove the following lemma.

LEMMA: The equation

$$A_\ell = A_m - A_n \tag{1}$$

is satisfied by the positive integers

$$\ell = (ra + 1)s \tag{2}$$

$$m = n + s = \frac{1}{2}\{(r^2a^2 + 2ra + 2)s + rb\} \tag{3}$$

$$n = \frac{1}{2}\{ra(ra + 2)s + rb\} \tag{4}$$

where r is any positive integer and s is any sufficiently large positive integer that is odd if both a and r are odd.

PROOF: From (1), we have

$$\ell(a\ell + b) = (m - n)(am + an + b).$$

Therefore, the integers ℓ, m, and n which satisfy the relations

$$\ell = c(m - n),$$

and

$$a\ell + b = \frac{1}{c}(am + an + b),$$

for any possible constant c, give a solution of (1). Solving for m and n, we have

$$m = \frac{1}{2}\left\{\frac{\ell}{c} + c\ell + \frac{b}{a}(c - 1)\right\} = \frac{1}{2}\{(ra + 1)^2 s + s + rb\}$$

$$n = \frac{1}{2}\left\{-\frac{\ell}{c} + c\ell + \frac{b}{a}(c - 1)\right\} = \frac{1}{2}\{(ra + 1)^2 s - s + rb\},$$

where $\ell = cs$, and $c = ra + 1$ are the defining equations for r and s. Equations (2), (3), and (4) follow immediately.

By observing Equation (4) and recalling that $a \equiv b \pmod 2$, we see that if r is any positive integer and s is any integer that is odd if both a and r are odd, which also satisfies

$$s > \max\left\{0, -\frac{b}{a(ra + 2)}\right\},$$

then ℓ, m, and n are positive integers, and the lemma is proved.

To prove the theorem, we first observe that for any t that satisfies the same condition as s in the lemma,

$$\ell' = \frac{1}{2}\{ra(ra + 2)t + rb\},$$

$$m' = \frac{1}{2}\{(r^2a^2 + 2ra + 2)t + rb\},$$

$$n' = (ra + 1)t,$$

satisfy the equation

$$A_{\ell'} = A_{m'} - A_{n'}. \tag{5}$$

Now we shall determine values of s and t so that we have $\ell = \ell'$. For these values, (1) and (5) will yield the required representations.

Let

$$s = \frac{1}{2}\{ra(ra + 2)x + r(ra + 1)b\}, \tag{6}$$

$$t = (ra + 1)x + rb, \tag{7}$$

where x is an integer that makes s odd if ra is odd. Then we have

$$\ell = (ra + 1)s = \frac{1}{2}\{ra(ra + 2)t + rb\} = \ell'$$

and thus, for x sufficiently large, s and t given by (6) and (7) will satisfy our requirement. Substituting (6) and (7) into ℓ, m, n, m', and n', we get the following proposition, which establishes the theorem.

PROPOSITION: If x is a sufficiently large integer that makes s in (6) *odd whenever ra is odd, then*

$$\ell = \frac{1}{2}\{ra(ra + 1)(ra + 2)x + r(ra + 1)^2b\}$$

$$m = \frac{1}{4}\{ra(ra + 2)(r^2a^2 + 2ra + 2)x + r(r^3a^3 + 3r^2a^2 + 4ra + 4)b\}$$

$$n = \frac{1}{4}\{r^2a^2(ra + 2)^2x + r(r^3a^3 + 3r^2a^2 + 2ra + 2)b\}$$

$$m' = \frac{1}{2}\{(ra + 1)(r^2a^2 + 2ra + 2)x + r(r^2a^2 + 2ra + 3)b\}$$

$$n' = (ra + 1)^2x + r(ra + 1)b$$

are positive integers, with $m \neq m'$, which satisfy the relation

$$A_\ell = A_m - A_n = A_{m'} - A_{n'}.$$

Note that for any r, a, and b, the equation $m = m'$ has at most one solution x, because it can be reduced to the equation

$$(r^2a^2 - 2)x = -r(ra - 1)b.$$

3. THE CASE OF POLYGONAL NUMBERS

If we put

$$a = k - 2,\ b = -(k - 4),\ \text{for } k \geqslant 3,$$

in ℓ, m, n, m', and n' in the proposition, we get formulas for polygonal numbers which satisfy the equation

$$P_{\ell,k} = P_{m,k} - P_{n,k} = P_{m',k} - P_{n',k}. \tag{8}$$

If $r = 1$, for instance, then we have

$$\ell = \frac{1}{2}\{k(k - 1)(k - 2)x - (k - 1)^2(k - 4)\}$$

$$m = \frac{1}{4}\{k(k - 2)(k^2 - 2k + 2)x - k(k - 4)(k^2 - 3k + 4)\}$$

$$n = \frac{1}{4}\{k^2(k - 2)^2x - (k - 4)(k^3 - 3k^2 + 2k + 2)\}$$

$$m' = \frac{1}{2}\{(k - 1)(k^2 - 2k + 2)x - (k - 4)(k^2 - 2k + 3)\}$$

$$n' = (k - 1)^2x - (k - 1)(k - 4).$$

For every positive integer x, if k is even, and for positive x such that $x \equiv k + 1 \pmod 4$, if k is odd, these values are positive integers with $m \neq m'$, which satisfy Equation (8).

In the case of $r = 2$ we have, for every positive integer x,

$$\ell = 2(k - 1)(k - 2)(2k - 3)x - (k - 4)(2k - 3)^2$$

$$m = 2(k - 1)(k - 2)(2k^2 - 6k + 5)x - 2(k - 4)(2k^3 - 9k^2 + 14k - 7)$$

$$n = 4(k - 1)^2(k - 2)^2x - (k - 4)(4k^3 - 18k^2 + 26k - 11)$$

$$m' = (2k - 3)(2k^2 - 6k + 5)x - (k - 4)(4k^2 - 12k + 11)$$

$$n' = (2k - 3)^2x - 2(k - 4)(2k - 3),$$

which are positive integers with $m \neq m'$, which satisfy Equation (8).

For $k = 3$ and 5, these values are as follows. In the case of $r = 1$, we use $4x$ for $k = 3$ and $4x - 2$ for $k = 5$ instead of x, so that we can get positive integral values for every positive integer x.

	$r = 1$	$r = 2$
$k = 3$	$\ell = 12x + 2$ $m = 15x + 3$ $n = 9x + 2$ $m' = 20x + 3$ $n' = 16x + 2$	$\ell = 12x + 9$ $m = 20x + 16$ $n = 16x + 13$ $m' = 15x + 11$ $n' = 9x + 6$
$k = 5$	$\ell = 120x - 68$ $m = 255x - 145$ $n = 225x - 128$ $m' = 136x - 77$ $n' = 64x - 36$	$\ell = 168x - 49$ $m = 600x - 176$ $n = 576x - 169$ $m' = 175x - 51$ $n' = 49x - 14$

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2. S. Ando. "A Note on the Polygonal Numbers." *The Fibonacci Quarterly* 19, No. 2 (1981):180-183.

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