ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each problem or solution should be on a separate sheet (or sheets). Preference will be given to those that are typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-484 Proposed by Philip L. Mana, Albuquerque, NM

For a given x, what is the least number of multiplications needed to calculate x^{98}? (Assume that storage is unlimited for intermediate products.)

B-485 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Find the complete solution u_n to the difference equation

$$u_{n+2} - 5u_{n+1} + 6u_n = 11F_n - 4F_{n+2}.$$

B-486 Proposed by Valentina Bakinova, Rondout Valley, NY

Prove or disprove that, for every positive integer k,

$$\frac{F_{k+1}}{F_1} < \frac{F_{k+3}}{F_3} < \frac{F_{k+5}}{F_5} < \cdots < a^k < \cdots < \frac{F_{k+6}}{F_6} < \frac{F_{k+4}}{F_4} < \frac{F_{k+2}}{F_2}.$$

B-487 Proposed by Herta T. Freitag, Roanoke, VA

Prove or disprove that, for all positive integers n,

$$5L_{4n} - L_{2n}^2 + 6 - 6(-1)^n L_{2n} \equiv 0 \pmod{10F_n^2}.$$

B-488 Proposed by Herta T. Freitag, Roanoke, VA

Let a and d be positive integers with d odd. Prove or disprove that for all positive integers h and k,

$$L_{a+hd} + L_{a+hd+d} \equiv L_{a+kd} + L_{a+kd+d} \pmod{L_d}.$$

B-489 Proposed by Herta T. Freitag, Roanoke, VA

Is there a Fibonacci analogue (or semianalogue) of B-488?

SOLUTIONS

Pythagorean Triples

B-457 Proposed by Herta T. Freitag, Roanoke, VA

Prove or disprove that there exists a positive integer b such that the Pythagorean-type relationship $(5F_n^2)^2 + b^2 \equiv (L_n^2)^2 \pmod{5m^2}$ holds for all m and n with $m|F_n$.

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We will show that the specified Pythagorean-type relationship holds with $b = 4$. Since

$$L_n^2 = 5F_n^2 + 4(-1)^n, \; (L_n^2)^2 = (5F_n^2)^2 + 8(-1)^n(5F_n^2) + 4^2,$$

we have

$$(5F_n^2)^2 + 4^2 \equiv (L_n^2)^2 \pmod{5F_n^2}.$$

Hence, for all m such that m divides F_n,

$$(5F_n^2)^2 + 4^2 \equiv (L_n^2)^2 \pmod{5m^2}.$$

Also solved by Paul S. Bruckman, Frank Higgins, Sahib Singh, Lawrence Somer, and the proposer.

Prime Difference of Triangular Numbers

B-458 Proposed by H. Klauser, Zurich, Switzerland

Let T_n be the triangular number $n(n+1)/2$. For which positive integers k do there exist positive integers n such that $T_{n+k} - T_n$ is a prime?

Solution by Lawrence Somer, Washington, D.C.

The answer is $k = 1$ or $k = 2$. Note that

$$T_{n+k} - T_n = (n + k)(n + k + 1)/2 - n(n + 1)/2$$
$$= (k^2 + k + 2nk)/2 = k(k + 2n + 1)/2.$$

If $T_{n+k} - T_n$ is prime, then $k = 1$ or $k/2 = 1$ since $k + 2n + 1 > k$. If $k = 1$, then $n = p - 1$, where p is prime, suffices to make $T_{n+k} - T_n$ prime. If $k = 2$, then $n = (p - 3)/2$, where p is prime, suffices to make $T_{n+k} - T_n$ prime.

Also solved by Paul Bruckman, Herta Freitag, Frank Higgins, Walther Janous, Peter Lindstrom, Bob Prielipp, Sahib Singh, J. Suck, Gregory Wulczyn, and the proposer.

Incongruent Differences

B-459 Proposed by E. E. McDonnell, Palo Alto, CA and J. O. Shallit, Berkeley, CA

Let g be a primitive root of the odd prime p. For $1 \leqslant i \leqslant p - 1$, let a_i be the integer in $S = \{0, 1, \ldots, p - 2\}$ with $g^{a_i} \equiv i \pmod{p}$. Show that

$$a_2 - a_1,\ a_3 - a_2,\ \ldots,\ a_{p-1} - a_{p-2}$$

(differences taken mod $p - 1$ to be in S), is a permutation of $1, 2, \ldots, p - 2$.

Solution by Lawrence Somer, Washington, D.C.

Suppose that $a_{i+1} - a_i \equiv a_{j+1} - a_j \pmod{p - 1}$, where $1 \leqslant i < j \leqslant p - 2$. Then

$$g^{a_{i+1} - a_i} \equiv g^{a_{j+1} - a_j} \pmod{p}$$

or

$$g^{a_{i+1}}/g^{a_i} \equiv (i + 1)/i \equiv g^{a_{j+1}}/g^{a_j} \equiv (j + 1)/j \pmod{p}.$$

Since neither i nor $j \equiv 0 \pmod{p}$, this implies that

$$(i + 1)j = ij + j \equiv i(j + 1) = ij + i \pmod{p}.$$

However, this is a contradiction, since $i \not\equiv j \pmod{p}$.

Also solved by Paul S. Bruckman, Frank Higgins, Walther Janous, Bob Prielipp, Sahib Singh, and the proposer.

First of a Pair

B-460 Proposed by Larry Taylor, Rego Park, NY

For all integers j, k, n, prove that

$$F_k F_{n+j} - F_j F_{n+k} = (-1)^j F_{k-j} F_n.$$

Solution by A. G. Shannon, New South Wales I.T., Australia

$$F_kF_{n+j} - F_jF_{n+k} = (a^k - b^k)(a^{n+j} - b^{n+j})/5 - (a^j - b^j)(a^{n+k} - b^{n+k})$$
$$= (ab)^j(a^{k-j} - b^{k-j})(a^n - b^n)/5$$
$$= (-1)^jF_{k-j}F_n.$$

Also solved by Clyde Bridger, Paul Bruckman, D. K. Chang, Herta Freitag, John Ivie, Walther Janous, John Milsom, Bob Prielipp, Heinz-Jurgen Seiffert, Sahib Singh, Gregory Wulczyn, and the proposer.

Companion Identity

B-461 Proposed by Larry Taylor, Rego Park, NY

For all integers j, k, n, prove or disprove that

$$F_kL_{n+j} - F_jL_{n+k} = (-1)^jF_{k-j}L_n.$$

Solution by Paul S. Bruckman, Sacramento, CA

The following relation follows readily from the Binet definitions:

$$F_uL_v = F_{v+u} - (-1)^uF_{v-u}. \quad (1)$$

Therefore,

$$F_kL_{n+j} - F_jL_{n+k} = F_{n+j+k} - (-1)^kF_{n+j-k} - F_{n+k+j} + (-1)^jF_{n+k-j}$$
$$= (-1)^j(F_{n+k-j} - (-1)^{k-j}F_{n-(k-j)})$$
$$= (-1)^jF_{k-j}L_n$$

[using (1) again, with $u = k - j$, $v = n$].

Also solved by Clyde Bridger, Herta Freitag, John Ivie, Walther Janous, John Milsom, Bob Prielipp, A. G. Shannon, Sahib Singh, Gregory Wulczyn, and the proposer.

Typographical Monstrosity

B-462 Proposed by Herta T. Freitag, Roanoke, VA

Let $L(n)$ denote L_n and $T_n = n(n + 1)/2$. Prove or disprove:

$$L(n) = (-1)^{T_{n-1}}[L(T_{n-1})L(T_n) - L(n^2)].$$

Solution by John W. Milsom, Butler County Community College, Butler, PA

Using $L(n) = L_n = a^n + b^n$, $ab = -1$, and $T_n = n(n + 1)/2$, it follows that

$$(-1)^{T_{n-1}}[L(T_{n-1})L(T_n) - L(n^2)] = (ab)^{n(n-1)}(a^n + b^n) = (-1)^{n(n-1)}L_n.$$

The number $n(n - 1)$ is always even, so that $(-1)^{n(n-1)} = 1$. Thus

$$L(n) = (-1)^{T_{n-1}}[L(T_{n-1})L(T_n) - L(n^2)].$$

Also solved by Clyde Bridger, Paul Bruckman, Walther Janous, Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

Casting Out Fives

B-463 Proposed by Herta T. Freitag, Roanoke, VA

Using the notations of B-462, prove or disprove:

$$L(n) \equiv (-1)^{T_{n-1}}L(n^2) \pmod 5.$$

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

We shall prove that the given congruence holds. Let $F(n)$ denote F_n. It is known that

$$L(a + b) - (-1)^b L(a - b) = 5F(a)F(b)$$

[see (10) and (12) on p. 115 of the April 1975 issue of this journal.] Hence,

$$L(T_n + T_{n-1}) - (-1)^{T_{n-1}}L(T_n - T_{n-1}) = 5F(T_n)F(T_{n-1})$$

so

$$L(n^2) - (-1)^{T_{n-1}}L(n) \equiv 0 \pmod 5.$$

The desired result follows almost immediately.

Also solved by Clyde Bridger, Paul Bruckman, Walther Janous, Sahib Singh, Gregory Wulczyn, and the proposer.

Consequence of a Hoggatt Identity

B-464 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Let n and w be integers with w odd. Prove or disprove:

$$F_{n+2w}F_{n+w} - 2L_wF_{n+w}F_{n-w} - F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

The given equation is equivalent to:

$$F_{n+2w}F_{n+w} - F_{n-w}F_{n-2w} - L_{3w}F_n^2 = 2L_w(F_{n+w}F_{n-w} - F_n^2).$$

Using I_{19} (*Fibonacci and Lucas Numbers* by Hoggatt), the right side

$$= 2(-1)^n L_w F_w^2.$$

Expressing the left side of the above equation in a and b, it simplifies to

$$\frac{2(-1)^n}{5}(L_{3w} + L_w) = 2(-1)^n L_w F_w^2.$$

Also solved by Paul Bruckman, Herta Freitag, Walther Janous, Bob Prielipp, M. Wachtel, and the proposer.

Evenly Proportioned

B-465 Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

For positive integers n and k, prove or disprove:

$$\frac{F_{2k} + F_{6k} + F_{10k} + \cdots + F_{(4n-2)k}}{L_{2k} + L_{6k} + L_{10k} + \cdots + L_{(4n-2)k}} = \frac{F_{2nk}}{L_{2nk}}.$$

Solution by Sahib Singh, Clarion State College, Clarion, PA

Expressing

$$F_{2k} = \frac{a^{2k} - b^{2k}}{\sqrt{5}} \quad \text{and} \quad L_{2k} = a^{2k} + b^{2k},$$

the left side of the equation simplifies to

$$\frac{F_{(4n+2)k} - F_{(4n-2)k} - 2F_{2k}}{L_{(4n+2)k} - L_{(4n-2)k}}$$

Using I_{24} and I_{16} (*Fibonacci and Lucas Numbers* by Hoggatt) successively, the above becomes

$$\frac{5F_{2k}F_{2nk}^2}{L_{(4n+2)k} - L_{(4n-2)k}}.$$

Since $L_{(4n+2)k} - L_{(4n-2)k} = 5F_{2k}F_{2nk}L_{2nk}$, we are done.

Also solved by Clyde Bridger, Paul Bruckman, Herta Freitag, Bob Prielipp, and the proposer.

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